COHOMOLOGY OF LOCAL COCHAINS

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Abstract. We prove that for generalised partitions of unity $\{\varphi_i \mid i \in I\}$ and coverings $\mathfrak{U} := \{\varphi_i^{-1}(R \setminus \{0\}) \mid i \in I\}$ of a topological space X the cohomology of abstract $\mathfrak{U}$-local cochains coincides with the cohomology of continuous $\mathfrak{U}$-local cochains, provided the coefficients are loop contractible. Furthermore we show that for each locally contractible group G and loop contractible coefficient group V the complex of germs of continuous functions on left-invariant diagonal neighbourhoods computes the Alexander-Spanier and singular cohomology; Similar results are obtained for k-groups and for germs of smooth functions on Lie groups.

Introduction

It is well known that the Alexander-Spanier cohomology $H_{AS}(X;V)$ of a topological space X with coefficients in a real topological vector space V coincides with its continuous version, provided the space X is paracompact. Analogously for smoothly paracompact manifolds M the Alexander-Spanier cohomology $H_{AS}(M;V)$ coincides with the smooth Alexander-Spanier cohomology $H_{AS,s}(M;V)$. The standard proof thereof uses sheaf theory and is not suited to show that one may in fact restrict oneself to diagonal neighbourhoods of a certain kind, e.g. left-invariant ones in topological groups. In the first part we give an alternate and more generally applicable proof which also demonstrates that for loop contractible coefficients and coverings $\mathfrak{U}$ of a topological space X by cozero sets of a generalised partition of unity the cohomology of abstract $\mathfrak{U}$-local cochains coincides with the cohomology of continuous $\mathfrak{U}$-local cochains, and for manifolds also coincides with the cohomology of smooth $\mathfrak{U}$-local cochains, if $\mathfrak{U}$ consists of cozero sets of a smooth generalised partition of unity and the coefficients are smoothly loop contractible. Passage to the colimit over all (numerable) coverings yields the classical result for paracompact spaces or smoothly paracompact manifolds.

We also relate the different cohomology concepts(Alexander-Spanier, singular and Čech cohomology) in locally contractible topological groups and Lie groups. Van Est has already shown in [vE62] that for locally contractible topological groups G on may compute one can compute the Alexander-Spanier cohomology by considering left-invariant neighbourhoods of the diagonals in G^{*+1} only. We extend his result to continuous and smooth cochains.

1. Local, Čech, Alexander-Spanier and Singular Cochains

Let X be a topological space and V be an abelian topological group. For each open covering $\mathfrak{U}$ of X and each $n \in \mathbb{N}$ one can define an open neighbourhood $\mathfrak{U}[n]$

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of the diagonal in X^{n+1} via

$$\mathfrak{U}[n] := \bigcup_{U \in \mathfrak{U}} U^{n+1} .$$

These neighbourhoods of the diagonals in X^{*+1} form an open simplicial subspace of X^{*+1} which allows us to consider complexes of cochains defined on them. We define abelian groups of n-cochains and continuous n-cochains with values in V:

$$A^n(\mathfrak{U}; V) := \{f : \mathfrak{U}[n] \to V\} \qquad \text{and} \qquad A_c^n(\mathfrak{U}; V) := C(\mathfrak{U}[n], V)$$

The elements of $A^n(\mathfrak{U}; V)$ and $A_c^n(\mathfrak{U}; V)$ are called $\mathfrak{U}$-*local* n-*cochains* and *continuous* $\mathfrak{U}$-*local* n-*cochains* respectively. The abelian groups $A^n(\mathfrak{U}; V)$ and $A_c^n(\mathfrak{U}; V)$ form cochain complexes with the usual differential given by

$$df(u_0, \ldots, u_{n+1}) := \sum_i (-1)^i f(u_0, \ldots, \hat{u}_i, \ldots, u_{n+1}) . \tag{1.1}$$

The cohomologies of these complexes are denoted by $H(\mathfrak{U}; V)$ and $H_c(\mathfrak{U}; V)$ respectively; they are called the $\mathfrak{U}$-*local cohomology* and the *continuous* $\mathfrak{U}$-*local cohomology*. The colimit complex $A^*_{AS}(X; V) := \operatorname{colim}_{\mathfrak{U}} A^*(\mathfrak{U}; V)$ where $\mathfrak{U}$ ranges over all open coverings of X is the complex of Alexander-Spanier cochains which computes the Alexander-Spanier cohomology $H_{AS}(X; V)$ of X. The cohomology of the continuous version $A^*_{AS,c}(X; V) := \operatorname{colim}_{\mathfrak{U}} A^*(\mathfrak{U}; V)$ is the continuous Alexander-Spanier cohomology $H_{AS,c}(X; V)$ of X.

We will show (in Section 2) that the inclusion $A_c^*(\mathfrak{U}; V) \hookrightarrow A^*(\mathfrak{U}; V)$ of cochain complexes induces an isomorphism in cohomology if $\mathfrak{U}$ is a covering by cozero sets of a generalised partition of unity. For this purpose we consider the Čech-Alexander-Spanier double complex $\check{C}^*(\mathfrak{U}, A^*)$ for open coverings $\mathfrak{U} = \{U_i \mid i \in I\}$ of X whose groups are given by

$$\check{C}^p(\mathfrak{U}, A^q) := \left\{ f \in \prod_{i_0, \ldots, i_p \in I} A^q(U_{i_0 \ldots i_p}; V) \mid \forall \sigma \in S_p : f_{i_0, \ldots, i_p} = \operatorname{sign}(\sigma) f_{i_{\sigma(0)} \ldots i_{\sigma(p)}} \right\}$$

and whose horizontal and vertical differentials d_h, d_v on $\check{C}^p(\mathfrak{U}, A^q)$ are given by the Čech coboundary operator δ and $(-1)^p$ times the products of the differentials d of the standard complexes $A^*(U_{i_0 \ldots i_p}; V)$ (cf. Eq. 1.1) respectively. The rows of the double complex $\check{C}^p(\mathfrak{U}, A^q)$ can be augmented by the complex $A^*(\mathfrak{U}; V)$ of $\mathfrak{U}$-local cochains and the columns can be augmented by the Čech-complex $\check{C}^*(\mathfrak{U}; V)$ for the covering $\mathfrak{U}$:

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \\
A^2(\mathfrak{U};V) \arrow[r] \arrow[u, "d_v"] & \check{C}^0(\mathfrak{U},A^2) \arrow[r, "d_h"] \arrow[u, "d_v"] & \check{C}^1(\mathfrak{U},A^2) \arrow[r, "d_h"] \arrow[u, "d_v"] & \check{C}^2(\mathfrak{U},A^2) \arrow[r, "d_h"] \arrow[u, "d_v"] & \cdots \\
A^1(\mathfrak{U};V) \arrow[r] \arrow[u, "d_v"] & \check{C}^0(\mathfrak{U},A^1) \arrow[r, "d_h"] \arrow[u, "d_v"] & \check{C}^1(\mathfrak{U},A^1) \arrow[r, "d_h"] \arrow[u, "d_v"] & \check{C}^2(\mathfrak{U},A^1) \arrow[r, "d_h"] \arrow[u, "d_v"] & \cdots \\
A^0(\mathfrak{U};V) \arrow[r] \arrow[u, "d_v"] & \check{C}^0(\mathfrak{U},A^0) \arrow[r, "d_h"] \arrow[u, "d_v"] & \check{C}^1(\mathfrak{U},A^0) \arrow[r, "d_h"] \arrow[u, "d_v"] & \check{C}^2(\mathfrak{U},A^0) \arrow[r, "d_h"] \arrow[u, "d_v"] & \cdots \\
 & \check{C}^0(\mathfrak{U};V) \arrow[r, "d_h"] \arrow[u] & \check{C}^1(\mathfrak{U};V) \arrow[r, "d_h"] \arrow[u] & \check{C}^2(\mathfrak{U};V) \arrow[r, "d_h"] \arrow[u] & \cdots
\end{tikzcd}$$

We denote the total complex of the double complex $\check{C}^*(\mathfrak{U},A^*)$ by $\mathrm{Tot}\check{C}^*(\mathfrak{U},A^*)$. The augmentations of the rows and columns of the double complex $\check{C}^*(\mathfrak{U},A^*)$ induce homomorphisms $i^*: A^*(\mathfrak{U};V) \to \mathrm{Tot}\check{C}^*(\mathfrak{U},A^*)$ and $j^*: \check{C}^*(\mathfrak{U};V) \to \mathrm{Tot}\check{C}^*(\mathfrak{U},A^*)$ of cochain complexes respectively.

Lemma 1.1. *The homomorphism $j^*: \check{C}^*(\mathfrak{U};V) \to \mathrm{Tot}\check{C}^*(\mathfrak{U},A^*)$ induces an isomorphism in cohomology.*

Proof. The augmented columns of the double complex $\mathrm{Tot}\check{C}^*(\mathfrak{U},A^*)$ are exact, because the standard complex $A^*(U_{i_0\ldots i_q};V)$ of a topological space $U_{i_0\ldots i_q}$ is always exact. Therefore the augmentation j^* induces an isomorphism in cohomology. $\square$

The augmented rows $A^q(\mathfrak{U};A) \hookrightarrow \check{C}^*(\mathfrak{U},A^q)$ are also exact; in fact, extending every function $f_{i_0\ldots i_p} \in A^q(U_{i_0\ldots i_p};V)$ to $U^q_{i_1\ldots i_p}$ by requiring it to be zero outside $U^q_{i_0\ldots i_p}$ we observe:

Proposition 1.2. *For any point finite set $\{\varphi_{q,i} \mid i \in I\}$ of (not necessarily continuous) $\mathbb{Z}$-valued functions on $\mathfrak{U}[q]$ satisfying $\sum_i \varphi_i = 1$ and $\varphi_{i|\mathfrak{U}[q]\setminus U_i^{q+1}} = 0$ the homomorphisms*

$$h^{p,q}: \check{C}^p(\mathfrak{U},A^q) \to \check{C}^{p-1}(\mathfrak{U},A^q), \quad h^{p,q}(f)_{i_0\ldots i_{p-1}} = \sum_i \varphi_{q,i} \cdot f_{ii_0\ldots i_{p-1}} \tag{1.2}$$

form a row contraction of the augmented row $A^q(\mathfrak{U};A) \hookrightarrow \check{C}^*(\mathfrak{U},A^q)$.

Proof. This is similar to the row contractions of the Čech-deRham complex of a manifold using smooth partitions of unity φ_i subordinate to $\mathfrak{U}$. For any cochain

$f \in \check{C}^p(\mathfrak{U}; A^q)$ of bidegree (p,q) the horizontal coboundary of $h^{p,q}(f)$ computes to

$$\begin{aligned}
(\delta h^{p,q}(f))_{i_0\dots i_p}(\vec{x}) &= \sum_{k=0}^{p}(-1)^k h^{p,q}(f)_{i_0\dots\hat{i}_k\dots i_p}(\vec{x}) \\
&= \sum_{k=0}^{p}(-1)^k \sum_i \varphi_{q,i}(\vec{x}) f_{ii_0\dots\hat{i}_k\dots i_{p-1}}(\vec{x}) \\
&= \sum_i \varphi_{q,i}(\vec{x}) \sum_{k=0}^{p}(-1)^k f_{ii_0\dots\hat{i}_k\dots i_{p-1}}(\vec{x}) \\
&= \sum_i \varphi_{q,i}(\vec{x}) \left[f_{i_0\dots i_p}(\vec{x}) - (\delta f)_{ii_0\dots i_p}(\vec{x})\right] \\
&= f_{i_0\dots i_p}(\vec{x}) - h^{p+1,q}(\delta f)_{ii_0\dots i_p}(\vec{x}) \,,
\end{aligned}$$

hence $h^{*,q}$ is a row contraction of the augmented row $A^q(\mathfrak{U}; A) \hookrightarrow \check{C}^*(\mathfrak{U}, A^q)$. □

Corollary 1.3. *For any open covering $\mathfrak{U} = \{U_i \mid i \in I\}$ of a topological space X the homomorphism $i^* : A^*(\mathfrak{U}; V) \to \mathrm{Tot}\check{C}^*(\mathfrak{U}, A^*)$ induces an isomorphism in cohomology.*

Proof. This follows from the fact that sets $\{\varphi_{q,i}\}$ of functions as required in Proposition 1.2 always exist: Well order the index set I and inductively define the functions $\varphi_{q,i}$ to be $\varphi_{q,i} := 1 - \max\{\varphi_j \mid j < i\}$ on U_i^{q+1} and zero on $X \setminus U_i$. □

Corollary 1.4. *For any open covering $\mathfrak{U}$ of a topological space X the Čech cohomology $\check{H}(\mathfrak{U}; V)$ for the covering $\mathfrak{U}$ and the cohomology $H(\mathfrak{U}; V)$ of $\mathfrak{U}$-local cochains are isomorphic. Thus the Čech Cohomology $\check{H}(\mathfrak{U}; V)$ can be computed from the complex $A^*(\mathfrak{U}; V)$ of $\mathfrak{U}$-local cochains.*

Example 1.5. If U is an open identity neighbourhood of a topological group G then the open covering $\mathfrak{U}_U := \{gU \mid g \in G\}$ of G is left invariant and the Čech Cohomology for the covering $\mathfrak{U}_U$ can be computed from the complex $A^*(\mathfrak{U}; V)$.

Passing to the colimit over all open coverings or all numerable open coverings yields the classical result:

Corollary 1.6 (Well known)**.** *For any topological space X the Čech cohomology $\check{H}(X; V)$ and the Alexander-Spanier cohomology $H_{AS}(X; V)$ are isomorphic. The same is true for the Čech cohomology w.r.t. numerable coverings and the Alexander-Spanier cohomology w.r.t. numerable coverings.*

If we replace the pre-sheaf $A^q(-; V)$ by the pre-sheaf $S^q(-; V)$ of singular q-cochains we obtain a double complex $\check{C}^p(\mathfrak{U}, S^q)$ for every open cover $\mathfrak{U}$ of X. The rows of this double complex can be augmented by the complex $S^*(\mathfrak{U}; V)$ of cochains on $\mathfrak{U}$-small singular simplices and the columns can be augmented by the Čech complex $\check{C}^p(\mathfrak{U}, V)$. However the augmented columns need not be exact; this only happens if each open set $U_{i_0\dots i_p}$ is V-acyclic, (i.e. it has trivial reduced singular cohomology with coefficients V), e.g. if each open set $U_{i_0\dots i_p}$ is contractible.

Lemma 1.7. *For any open covering $\mathfrak{U} = \{U_i \mid i \in I\}$ of X for which the sets $U_{i_0\dots i_p}$ are V-acyclic the Čech cohomology $\check{H}(\mathfrak{U}; V)$ for the covering $\mathfrak{U}$ and the singular cohomology $H_{sing}(X; V)$ are isomorphic. In particular the Čech cohomology does not depend on the open cover subject to the acyclicity condition chosen.*

Proof. If each reduced singular cohomology $\tilde{H}_{sing}(U_{i_0\dots i_p};V)$ is trivial then the augmented columns of the double complex $\check{C}^p(\mathfrak{U},S^q)$ are exact. Proceeding as for local cochains yields an isomorphism $H(S^*(\mathfrak{U};V)) \cong \check{H}(\mathfrak{U};V)$. The cohomology of the complex $S^*(\mathfrak{U};V)$ is the singular cohomology $H_{sing}(X;V)$ of X. $\square$

Example 1.8. If $\mathfrak{U}$ is a 'good' cover of a topological space X then the Čech cohomology $\check{H}(\mathfrak{U};V)$ for the covering $\mathfrak{U}$ is isomorphic to the singular cohomology $H_{sing}(X;V)$ of X.

Example 1.9. If $\mathfrak{U}$ is an open covering of a finite dimensional Riemannian manifold M by geodetically convex sets, then the Čech cohomology $\check{H}(\mathfrak{U};V)$ for the covering $\mathfrak{U}$ is isomorphic to the singular cohomology $H_{sing}(M;V)$ of M. If M is an infinite dimensional Riemannian manifold one has to require the local existence of geodesics for this argument to be applicable.

Example 1.10. If G is a Hilbert Lie group (i.e. a Lie group whose model space is a Hilbert space[1]), and U a geodetically convex identity neighbourhood of G, then the the Čech cohomology $\check{H}(\mathfrak{U}_U;V)$ for the covering $\mathfrak{U}_U := \{gU \mid g \in G\}$ is isomorphic to the singular cohomology $H_{sing}(G;V)$ of G.

In this case the singular and $\mathfrak{U}$-local cohomologies also coincide. We assert that this isomorphism is induced by a natural morphism $\lambda^* : A^*(\mathfrak{U};V) \to S^*(X,\mathfrak{U};V)$ of cochain complexes whose construction we briefly recall: Consider the singular semi-simplicial space $C(\Delta,X)$ of X and the vertex morphism $\lambda_X : C(\Delta,X) \to X^{*+1}$ of semi-simplicial spaces, which assigns to each singular n-simplex $\tau : \Delta^n \to X$ its ordered set of vertices $(\tau(\vec{e}_0),\dots,\tau(\vec{e}_n))$. This morphism λ induces a morphism $C(\lambda^*_{\mathfrak{U}},V) : A^*(\mathfrak{U};V) \to S^*(X,\mathfrak{U};V)$ of cochain complexes.

Proposition 1.11. *For any open covering $\mathfrak{U} = \{U_i \mid i \in I\}$ of X for which the sets $U_{i_0\dots i_p}$ are V-acyclic the morphism $C(\lambda^*_{\mathfrak{U}},V) : A^*(\mathfrak{U};V) \to S^*(X,\mathfrak{U};V)$ induces an isomorphism $H(\mathfrak{U};V) \cong H_{sing}(X;V)$ in cohomology. In particular the Čech and $\mathfrak{U}$-local cohomology do not depend on the open cover subject to the acyclicity condition chosen.*

Proof. Let $\mathfrak{U}$ be an open covering of X satisfying $\tilde{H}_{sing}(U_{i_0\dots i_p};V) = 0$ for all $i_0\dots i_p \in I$. The morphism $\lambda_X : C(\Delta,X) \to X^{*+1}$ of semi-simplicial not only induces a morphism $C(\lambda^*_{\mathfrak{U}},V) : A^*(\mathfrak{U};V) \to S^*(X,\mathfrak{U};V)$ of cochain complexes but also a morphism $\check{C}^*(\mathfrak{U};C(\lambda^*;V)) : \check{C}^*(\mathfrak{U};A^*) \to \check{C}^*(\mathfrak{U};S^*)$ of double complexes. The morphisms $C(\lambda^*_{\mathfrak{U}},V)$ and $\check{C}^*(\mathfrak{U};C(\lambda^*;V))$ intertwine the augmentations of the double complexes $\check{C}^*(\mathfrak{U};A^*)$ and $\check{C}^*(\mathfrak{U};S^*)$, leading to the commutative diagram

$$\begin{tikzcd}
A^*(\mathfrak{U};V) \arrow[r, "i^*"] \arrow[d, "C(\lambda^*_{\mathfrak{U}};V)"'] & \mathrm{Tot}\check{C}^*(\mathfrak{U},A^*) \arrow[d, "\check{C}^*(\mathfrak{U};C(\lambda^*;V))"] & \check{C}(\mathfrak{U};V) \arrow[l, "j^*"'] \arrow[d, equal] \\
S^*(\mathfrak{U};V) \arrow[r, "i^*"] & \mathrm{Tot}\check{C}^*(\mathfrak{U},S^*) & \check{C}(\mathfrak{U};V) \arrow[l, "j^*"']
\end{tikzcd}$$

of cochain complexes, in which all but the left downward morphisms induce isomorphisms in cohomology. Thus the morphism $C(\lambda^*_{\mathfrak{U}},V) : A^*(\mathfrak{U};V) \to S^*(X,\mathfrak{U};V)$ of cochain complexes induces an isomorphism in cohomology as well. $\square$

[1]We do not restrict ourselves to finite dimensional Lie groups.

Corollary 1.12. *For any open covering $\mathfrak{U} = \{U_i \mid i \in I\}$ of X for which the sets $U_{i_0 \dots i_p}$ are V-acyclic the singular cohomology of X and the Čech cohomology for the covering $\mathfrak{U}$ can be computed from the complex $A^*(\mathfrak{U}; V)$ of $\mathfrak{U}$-local cochains.*

Example 1.13. For any good covering $\mathfrak{U}$ of a topological space X the morphism $C(\lambda^*_{\mathfrak{U}}, V) : A^*(\mathfrak{U}; V) \to S^*(X, \mathfrak{U}; V)$ of cochain complexes induces an isomorphism in cohomology and the cohomologies $\check{H}(\mathfrak{U}; V)$, $H(\mathfrak{U}; V)$ and $H_{sing}(X; V)$ are isomorphic.

Lemma 1.14. *If the open coverings $\mathfrak{U}$ a topological space X for which the sets $U_{i_0 \dots i_p}$ are V-acyclic are cofinal in all open coverings, then for each such covering $\mathfrak{U}$ the Čech cohomology $\check{H}(\mathfrak{U}; V)$ for the covering $\mathfrak{U}$ coincides with the Čech cohomology $\check{H}(X; V)$ of X and the cohomology $H(\mathfrak{U}; V)$ of $\mathfrak{U}$-local cochains coincides with the Alexander-Spanier cohomology $H_{AS}(X; V)$ of X. In particular the systems $H(\mathfrak{U}; V)$ and $\check{H}(\mathfrak{U}; V)$ of abelian groups are co-Mittag-Leffler.*

Proof. If the open coverings $\mathfrak{U}$ satisfying $\tilde{H}_{sing}(U_{i_0 \dots i_p}; V) = 0$ for all $i_0 \dots i_p \in I$ of a space X are cofinal in all open coverings, then the Alexander-Spanier and Čech cohomologies of X can be computed as the colimit over all such open covers $\mathfrak{U}$, hence $H_{AS}(X; V) = \operatorname{colim}_{\mathfrak{U}} H(\mathfrak{U}; V) \cong \operatorname{colim}_{\mathfrak{U}} H_{sing}(X; V) = H_{sing}(X; V)$ and $\check{H}(X; V) = \operatorname{colim}_{\mathfrak{U}} \check{H}(\mathfrak{U}; V) \cong \operatorname{colim}_{\mathfrak{U}} H_{sing}(X; V) = H_{sing}(X; V)$, where the colimits are taken over all open coverings $\mathfrak{U}$ subject to the condition $\tilde{H}_{sing}(U_{i_0 \dots i_p}; V) = 0$ for all $i_0 \dots i_p \in I$. □

Example 1.15. Every covering of a finite dimensional Riemannian manifold M admits an open refinement $\mathfrak{U}$ by geodetically convex sets. Thus for any open covering $\mathfrak{U}$ of a finite dimensional Riemannian manifold M by geodetically convex sets the cohomology of the complexes $A^*(\mathfrak{U}; V)$ and $\check{C}^*(\mathfrak{U}; V)$ are isomorphic to the Alexander-Spanier, Čech and singular cohomologies of M. If M is an infinite dimensional Riemannian manifold one has to require the local existence of geodesics for this argument to be applicable.

Example 1.16. If G is a Hilbert Lie group and U a geodetically convex identity neighbourhood of G, then then the cohomology of the complexes $A^*(\mathfrak{U}_U; V)$ and $\check{C}^*(\mathfrak{U}; V)$ are both isomorphic to $H_{AS}(G; V)$, $\check{H}(\mathfrak{U}; V)$ and $H_{sing}(G; V)$.

For some topological spaces the acyclicity condition on the covering $\mathfrak{U}$ is not necessary to obtain similar results. For topological groups one can also consider the complexes $\operatorname{colim}_{U \in \mathcal{U}_1} A^*(\mathfrak{U}_U; V)$ and $\operatorname{colim}_{U \in \mathcal{U}_1} A^*_c(\mathfrak{U}_U; V)$, where U ranges over all open identity neighbourhoods of G. For these colimit complexes we observe:

Theorem 1.17. *For any locally contractible topological group G with open identity neighbourhood filterbase $\mathcal{U}_1$ the morphisms $C(\lambda^*_{\mathfrak{U}_U}, V) : A^*(\mathfrak{U}_U; V) \to S^*(\mathfrak{U}; V)$ induce an isomorphism $\operatorname{colim}_{U \in \mathcal{U}_1} H(\mathfrak{U}_U; V) \cong H_{sing}(X; V)$ in cohomology and the cohomologies $\operatorname{colim}_{U \in \mathcal{U}_1} H(\mathfrak{U}_U; V)$, $\operatorname{colim}_{U \in \mathcal{U}_1} \check{H}(\mathfrak{U}_U; V)$, and $H_{sing}(G; V)$ coincide.*

Proof. The first statement has been shown by van Est in [vE62]. The other isomorphisms then are a consequence of Corollary 1.4. □

Corollary 1.18. *For Lie groups G with open identity neighbourhood filterbase $\mathcal{U}_1$ the cohomologies $\operatorname{colim}_{U \in \mathcal{U}_1} H(\mathfrak{U}_U; V)$, $\operatorname{colim}_{U \in \mathcal{U}_1} \check{H}(\mathfrak{U}_U; V)$ and $H_{sing}(G; V)$ are isomorphic and the isomorphism $\operatorname{colim}_{U \in \mathcal{U}_1} H_c(\mathfrak{U}_U; V) \cong H_{sing}(G; V)$ is induced by the vertex morphism λ.*

2. Continuous Local Cochains

The preceding observations can – in parts – be generalised for continuous cochains. Replacing the pre-sheaf $A^q(-;V)$ of q-cochains by the pre-sheaf $A_c^q(-;V)$ of continuous q-cochains we obtain a sub double complex $\check{C}^*(\mathfrak{U}, A_c^*)$ of $\check{C}^*(\mathfrak{U}, A^*)$ whose groups are given by

$$\check{C}^p(\mathfrak{U}, A_c^q) := \left\{ f \in \check{C}^p(\mathfrak{U}, A^q) \mid \forall i_0, \dots i_p \in I : f_{i_0 \dots i_p} \in C(U^{q+1}_{i_0 \dots i_p}; V) \right\} .$$

The rows of this sub double complex $\check{C}^*(\mathfrak{U}, A_c^*)$ can be augmented by the complex $A_c^*(\mathfrak{U}; V)$ of continuous $\mathfrak{U}$-local cochains and the columns can be augmented by the Čech-complex $\check{C}^*(\mathfrak{U}; V)$ for the covering $\mathfrak{U}$. These augmentations induce homomorphisms $i_c^* : A_c^*(\mathfrak{U}; V) \to \mathrm{Tot}\check{C}^*(\mathfrak{U}, A_c^*)$ and $j_c^* : \check{C}^*(\mathfrak{U}; V) \to \mathrm{Tot}\check{C}^*(\mathfrak{U}, A_c^*)$ of cochain complexes respectively.

Lemma 2.1. *The homomorphism $j_c^* : \check{C}^*(\mathfrak{U}; V) \to \mathrm{Tot}\check{C}^*(\mathfrak{U}, A_c^*)$ induces an isomorphism in cohomology.*

Proof. The proof is analogous to that of Lemma 1.1. □

To obtain exact rows in the double complex $\check{C}^*(\mathfrak{U}, A_c^*)$ we impose a restriction on the coefficient group V. For modules V over some unital topological ring R one can replace the set $\{\varphi_{q,i}\}$ of functions in Proposition 1.2 by generalised partitions of unity to obtain results analogous to those in Section 1, as will be shown below. In the following the coefficients V will always be a topological module over a unital topological ring R. (Different coefficient groups are considered in the next section.)

Definition 2.2. A *generalised R-valued partition of unity* $\{\varphi_i \mid i \in I\}$ on a topological space X is a set of continuous functions $\varphi_i : X \to R$ which satisfies the equation $\sum_{i \in I} \varphi_i = 1$. An *$R$-valued partition of unity* $\{\varphi_i \mid i \in I\}$ on a topological space X is a generalised R-valued partition of unity whose supports $\operatorname{supp} \varphi_i = \overline{\varphi_i^{-1}(R \setminus \{0\})}$ form a locally finite covering of X.

Remark 2.3. For $R = \mathbb{R}$ one can always replace the functions φ_i by positive ones with the same zero set (cf. Lemma A.3).

For any R-valued partition of unity $\{\varphi_{q,i} \mid i \in I\}$ subordinate to the open covering $\{U_i^{q+1} \mid i \in I\}$ of $\mathfrak{U}[q]$ and continuous cochain $f \in \check{C}^p(\mathfrak{U}; A_c^q)$ the products $\varphi_{q,i} f_{ii_0 \dots i_{p-1}}$ have supports in the open sets $U^{q+1}_{ii_0 \dots i_p}$ respectively. Therefore they can be continuously extended to $U^{q+1}_{i_0 \dots i_{p-1}}$ by defining it to be zero outside $U^{q+1}_{ii_0 \dots i_{p-1}}$. Understanding each function $\varphi_{q,i} f_{ii_0 \dots i_{p-q}}$ to be extended this way we define a homotopy operator for continuous cochains:

Proposition 2.4. *For any R-valued partition of unity $\{\varphi_{q,i} \mid i \in I\}$ subordinate to the covering $\{U_i^{q+1} \mid i \in I\}$ of $\mathfrak{U}[q]$ the homomorphisms*

$$h^{p,q} : \check{C}^p(\mathfrak{U}, A^q) \to \check{C}^{p-1}(\mathfrak{U}, A^q), \quad h^{p,q}(f)_{i_0 \dots i_{p-1}} = \sum_i \varphi_{q,i} \cdot f_{ii_0 \dots i_{p-1}} \tag{2.1}$$

form a row contraction of the augmented row $A^q(\mathfrak{U}; V) \hookrightarrow \check{C}^*(\mathfrak{U}, A^q)$ which restricts to a row contraction of the augmented sub-row $A_c^q(\mathfrak{U}; V) \hookrightarrow \check{C}^*(\mathfrak{U}, A_c^q)$.

Proof. By Proposition 1.2 the maps $h^{p,q}$ form a row contraction of the augmented row $A^q(\mathfrak{U};V) \hookrightarrow \check{C}^*(\mathfrak{U},A^q)$. In addition they map continuous cochains to continuous cochains by construction, hence they restrict to a row contraction of the augmented sub-row $A^q_c(\mathfrak{U};V) \hookrightarrow \check{C}^*(\mathfrak{U},A^q_c)$. □

Definition 2.5. A covering $\mathfrak{U}$ of a topological space X is called R-numerable, if there exists an R-valued partition of unity subordinate to $\mathfrak{U}$.

Corollary 2.6. *For any open covering $\mathfrak{U} = \{U_i \mid i \in I\}$ of a topological space X for which the coverings $\{U_i^{q+1} \mid i \in I\}$ of the spaces $\mathfrak{U}[q]$ are R-numerable the homomorphism $i^*_c : A^*_c(\mathfrak{U};V) \to \mathrm{Tot}\check{C}^*(\mathfrak{U},A^*_c)$ induces an isomorphism in cohomology.*

Theorem 2.7. *For any open covering $\mathfrak{U}$ of a topological space X for which each covering $\{U_i^{q+1} \mid i \in I\}$ of $\mathfrak{U}[q]$ is R-numerable the inclusion $A^*_c(\mathfrak{U};V) \hookrightarrow A^*(\mathfrak{U};V)$ induces an isomorphism in cohomology and the cohomologies $\check{H}(\mathfrak{U};V)$, $H_c(\mathfrak{U};V)$ and $H(\mathfrak{U};V)$ are isomorphic.*

Proof. The inclusions $A_c(\mathfrak{U};V) \hookrightarrow A(\mathfrak{U};V)$ and $\mathrm{Tot}\check{C}^*(\mathfrak{U};A^*_c) \hookrightarrow \mathrm{Tot}\check{C}^*(\mathfrak{U};A^*)$ intertwine the augmentations i^*_c and i^*. Thus one obtains the following commutative diagram

$$\begin{tikzcd}
A^*_c(\mathfrak{U};V) \arrow[r, "i^*_c"] \arrow[d] & \mathrm{Tot}\check{C}^*(\mathfrak{U},A^*_c) \arrow[d] & \check{C}(\mathfrak{U};V) \arrow[l, "j^*_c"'] \arrow[d, equal] \\
A^*(\mathfrak{U};V) \arrow[r, "i^*"] & \mathrm{Tot}\check{C}^*(\mathfrak{U},A^*) & \check{C}(\mathfrak{U};V) \arrow[l, "j^*"']
\end{tikzcd}$$

where the horizontal arrows are induced by the augmentations and the vertical arrows are induced by inclusion. The homomorphisms j^*_c and j^* induces isomorphisms in cohomology by Lemmata 1.1 and 2.1. The homomorphisms i^* always induces an isomorphism as observed in Corollary 1.3. If the open coverings $\{U_i^{q+1} \mid i \in I\}$ of the spaces $\mathfrak{U}[q]$ are R-numerable, then the homomorphism i^*_c also induces an isomorphism in cohomology by Corollary 2.6. All in all we obtain the following commutative diagram

$$\begin{tikzcd}
H_c(\mathfrak{U};V) \arrow[r, "\cong"] \arrow[d] & H(\mathrm{Tot}\check{C}^*(\mathfrak{U},A^*_c)) \arrow[d] & \check{H}(\mathfrak{U};V) \arrow[l, "\cong"] \arrow[d, equal] \\
H(\mathfrak{U};V) \arrow[r, "\cong"] & H(\mathrm{Tot}\check{C}^*(\mathfrak{U},A^*)) & \check{H}(\mathfrak{U};V) \arrow[l, "\cong"]
\end{tikzcd}$$

in which all horizontal arrows and the right vertical arrow are isomorphisms. This forces the homomorphism $H_c(\mathfrak{U};V) \to H(\mathfrak{U};V)$ induced by inclusion to be an isomorphism as well. □

In this case the Čech Cohomology $\check{H}(\mathfrak{U};V)$ for the covering $\mathfrak{U}$ of X can be either computed from the complex $A^*_c(\mathfrak{U};V)$ of continuous $\mathfrak{U}$-local cochains or from from the complex $A^*(\mathfrak{U};V)$ of $\mathfrak{U}$-local cochains.

Corollary 2.8. *For $R = \mathbb{R}$, any generalised partition of unity $\{\varphi_i \mid i \in I\}$ on X and $\mathfrak{U} := \{\varphi_i^{-1}(R \setminus \{0\}) \mid i \in I\}$ the inclusion $A^*_c(\mathfrak{U};V) \hookrightarrow A^*(\mathfrak{U};V)$ induces an isomorphism in cohomology and the cohomologies $\check{H}(\mathfrak{U};V)$, $H_c(\mathfrak{U};V)$ and $H(\mathfrak{U};V)$ are isomorphic.*

Proof. In view of Theorem 2.7 it suffices to show that the coverings $\{U_i^{q+1} \mid i \in I\}$ of the spaces $\mathfrak{U}[q]$ are numerable. On each space $\mathfrak{U}[q]$ we can (by Lemmata A.4 and A.3) define non-negative continuous functions $\varphi_{q,i}$ and φ_q via

$$\begin{aligned} \varphi_{q,i} : \mathfrak{U}[q] \to \mathbb{R} \qquad & \varphi_{q,i}(\vec{x}) = |\varphi_{q,i}(x_0) \cdots \varphi_{q,i}(x_q)| \\ \varphi_q : \mathfrak{U}[q] \to \mathbb{R} \qquad & \varphi_q(\vec{x}) = \sum_i \varphi_{q,i}(\vec{x})\,. \end{aligned}$$

Since the functions φ_q are strictly non-zero on $\mathfrak{U}[q]$, the set $\{\varphi_q^{-1}\varphi_{q,i} \mid i \in I\}$ is a generalised partition of unity with cozero sets $\{U_i^{q+1} \mid i \in I\}$. Therefore the open covering $\{U_i^{q+1} \mid i \in I\}$ of $\mathfrak{U}[q]$ is numerable. □

The colimit over all R-numerable coverings $\mathfrak{U}$ is called the *Alexander-Spanier cohomology w.r.t R-numerable coverings.* Passing to the colimit over all R-numerable covers we observe:

Corollary 2.9. *The cohomologies $\check{H}(X;V)$, $H_{AS,c}(X;V)$ and $H_{AS}(X;V)$ of a topological space X w.r.t. R-numerable coverings are isomorphic.*

Calling a topological space *R-paracompact*, if every open covering $\mathfrak{U}$ of X admits an R-valued partition of unity subordinate to $\mathfrak{U}$ we also note:

Corollary 2.10. *The cohomologies $\check{H}(X;V)$, $H_{AS,c}(X;V)$ and $H_{AS}(X;V)$ of an R-paracompact topological space X are isomorphic.*

These observations can especially be applied to uniform spaces (e.g. topological groups) X with open coverings of the form $\mathfrak{U}_U := \{U[x] \mid x \in X\}$, where U is an open entourage of the diagonal in $X \times X$.

Proposition 2.11. *If $d : X \times X \to \mathbb{R}$ is a continuous pseudometric on X and V a real vector space, then for each $\epsilon > 0$ and covering $\mathfrak{U} = \{B_d(x,\epsilon) \mid x \in X\}$ of X by open ϵ-balls the inclusion $A_c^*(\mathfrak{U};V) \hookrightarrow A^*(\mathfrak{U};V)$ induces an isomorphism in cohomology and the cohomologies $\check{H}(\mathfrak{U};V)$, $H_c(\mathfrak{U};V)$ and $H(\mathfrak{U};V)$ are isomorphic.*

Proof. Let $d : X \times X \to \mathbb{R}$ be a continuous pseudometric on X and V be a real vector space. By [Seg70, Proposition B2] there exists a generalised partition of unity $\{\varphi_x \mid x \in X\}$ on X satisfying $\varphi_x^{-1}((0,1]) = B_d(x,\epsilon)$, so Corollary 2.8 applies. □

Example 2.12. If $\mathfrak{U} = \{B(x,\epsilon) \mid x \in X\}$ is an open covering of a finite dimensional Riemannian manifold M by open ϵ-balls and V a real topological vector space, then the cohomology $H_c(\mathfrak{U};V)$ of the complex $A_c^*(\mathfrak{U};V)$ is isomorphic to cohomology $H(\mathfrak{U};V)$ and to the Čech and singular cohomologies of M (cf. Ex 1.15). If M is an infinite dimensional Riemannian manifold one has to require the local existence of geodesics.

Corollary 2.13. *For any open entourage U of a uniform space X and real topological vector space V the inclusion $A_c^*(\mathfrak{U}_U;V) \hookrightarrow A^*(\mathfrak{U}_U;V)$ induces an isomorphism in cohomology and the cohomologies $\check{H}(\mathfrak{U}_U;V)$, $H_c(\mathfrak{U}_U;V)$ and $H(\mathfrak{U}_U;V)$ are isomorphic.*

Proof. This follows from Corollary 2.11 and the fact that open entourages of uniform spaces are always of the form $U = d_U^{-1}([0,1))$ for a continuous pseudometric $d_U : X \times X \to \mathbb{R}$ (cf. [Seg70, Proposition B.2]). □

A particular interesting case are topological groups with open coverings of the form $\mathfrak{U}_U := \{gU \mid g \in G\}$, where U is an open identity neighbourhood in G. Here the complexes $A_c^*(\mathfrak{U};V)$ and $A^*(\mathfrak{U};V)$ are sometimes called the complexes of *continuous U-local cochains* and *U-local cochains*. For this special case we observe:

Corollary 2.14. *For any open identity neighbourhood U of a topological group G and any real topological vector space V the inclusion $A_c^*(\mathfrak{U}_U;V) \hookrightarrow A^*(\mathfrak{U}_U;V)$ induces an isomorphism in cohomology and the cohomologies $\check{H}(\mathfrak{U}_U;V)$, $H_c(\mathfrak{U}_U;V)$ and $H(\mathfrak{U}_U;V)$ are isomorphic.*

Combining these results with those concerning singular cohomology (obtained in Section 1) we observe:

Proposition 2.15. *For any open covering $\mathfrak{U}$ of a topological space X for which each set $U_{i_0\dots i_p}$ is V-acyclic and each covering $\{U_i^{q+1} \mid i \in I\}$ of $\mathfrak{U}[q]$ is R-numerable the homomorphism $C(\lambda_{\mathfrak{U}}^*, V) : A_c^*(\mathfrak{U};V) \to S^*(X,\mathfrak{U};V)$ induces an isomorphism $H_c(\mathfrak{U};V) \cong H_{sing}(X;V)$ in cohomology and the following diagram is commutative:*

$$\begin{tikzcd}
H_c(\mathfrak{U};V) \arrow[r, "H(i)", "\cong"'] \arrow[d, "H(C(\lambda_{\mathfrak{U}}^*;V))"', "\cong"] & H(\mathfrak{U};V) \arrow[r, "H(j)^{-1}H(i)", "\cong"'] \arrow[d, "H(C(\lambda_{\mathfrak{U}}^*;V))"', "\cong"] & \check{H}(\mathfrak{U};V) \arrow[d, equal] \\
H_{sing}(\mathfrak{U};V) \arrow[r, equal] & H_{sing}(\mathfrak{U};V) \arrow[r, "\cong", "H(j)^{-1}H(i)"'] & \check{H}(\mathfrak{U};V)
\end{tikzcd}$$

In particular the Čech and the continuous $\mathfrak{U}$-local cohomology do not depend on the open cover $\mathfrak{U}$ subject to the above conditions chosen.

Corollary 2.16. *For any open covering $\mathfrak{U}$ of a topological space X for which each set $U_{i_0\dots i_p}$ is V-acyclic and each covering $\{U_i^{q+1} \mid i \in I\}$ of $\mathfrak{U}[q]$ is R-numerable the singular cohomology $H_{sing}(X;V)$ and the Čech cohomology $\check{H}(\mathfrak{U};V)$ for the covering $\mathfrak{U}$ can be computed from the complex $A^*(\mathfrak{U};V)$ of $\mathfrak{U}$-local cochains.*

Example 2.17. For any 'good' cover $\mathfrak{U}$ of a topological space X for which each covering $\{U_i^{q+1} \mid i \in I\}$ of $\mathfrak{U}[q]$ is R-numerable the morphism $A_c^*(\mathfrak{U};V) \to S^*(X,\mathfrak{U};V)$ of cochain complexes induces an isomorphism in cohomology and the cohomologies $\check{H}(\mathfrak{U};V)$, $H(\mathfrak{U};V)$, $H_c(\mathfrak{U};V)$ and $H_{sing}(X;V)$ are isomorphic.

Lemma 2.18. *If the open coverings $\mathfrak{U}$ of a topological space X for which the sets $U_{i_0\dots i_p}$ are V-acyclic and each covering $\{U_i^{q+1} \mid i \in I\}$ of $\mathfrak{U}[q]$ is R-numerable are cofinal in all open coverings, then for each such covering $\mathfrak{U}$ the the cohomology $H_c(\mathfrak{U};V)$ of continuous $\mathfrak{U}$-local cochains coincides with the continuous Alexander-Spanier cohomology $H_{AS,c}(X;V)$ of X. In particular the directed system $H_c(\mathfrak{U};V)$ of abelian groups is co-Mittag-Leffler.*

Proof. In this case the (continuous) Alexander-Spanier cohomology can be computed as the colimit over this cofinal set of open coverings $\mathfrak{U}$. Proposition 2.15 shows the isomorphisms $H_c(\mathfrak{U};V) \cong H(\mathfrak{U};V) \cong H_{sing}(X;V)$ for every covering $\mathfrak{U}$ in this cofinal set; this implies the isomorphism $H_{AS,c}(X;V) \cong H_{AS}(X;V)$ of the colimit groups. It also shows that the directed systems $H_c(\mathfrak{U};V)$ and $H(\mathfrak{U};V)$ are co-Mittag-Leffler. □

Example 2.19. If $\mathfrak{U}$ is an open covering of a finite dimensional Riemannian manifold M by geodetically convex sets, then the cohomology of the complex $A_c^*(\mathfrak{U};V)$

is isomorphic to the Čech cohomology $\check{H}(\mathfrak{U};V)$ and to the singular cohomology $H_{sing}(M;V)$ of M. If M is an infinite dimensional Riemannian manifold one has to require the local existence of geodesics for this argument to be applicable.

Example 2.20. If G is a Hilbert Lie group and U a geodetically convex identity neighbourhood of G, then the cohomology of the complex $A_c^*(\mathfrak{U}_U;V)$ is isomorphic to Čech cohomology $\check{H}(\mathfrak{U};V)$ and to the singular cohomology $H_{sing}(M;V)$ of M.

As observed before, one can obtain a similar result for locally contractible topological groups without acyclicity condition on the open coverings:

Theorem 2.21. *For any locally contractible group G with open neighbourhood filterbase $\mathcal{U}_1$ and real vector space V the morphisms $A_c^*(\mathfrak{U}_U;V) \hookrightarrow A^*(\mathfrak{U}_U;V)$ and $C(\lambda^*_{\mathfrak{U}_U},V) : A_c^*(\mathfrak{U}_U;V) \to S^*(\mathfrak{U}_U;V)$ for all $U \in \mathcal{U}_1$ induce isomorphisms*

$$\operatorname{colim}_{U\in\mathcal{U}_1} H_c(\mathfrak{U}_U;V) \cong \operatorname{colim}_{U\in\mathcal{U}_1} H(\mathfrak{U}_U;V) \cong H_{sing}(G;V)\,.$$

in cohomology.

Proof. The first statement is a consequence of Corollary 2.14. The other isomorphisms follow from Theorem 1.17. □

Corollary 2.22. *For Lie groups G with open identity neighbourhood filter $\mathcal{U}_1$ and real vector space V the cohomologies $\operatorname{colim}_{U\in\mathcal{U}_1} H_c(\mathfrak{U}_U;V)$, $\operatorname{colim}_{U\in\mathcal{U}_1} H(\mathfrak{U}_U;V)$, $\operatorname{colim}_{U\in\mathcal{U}_1} \check{H}(\mathfrak{U}_U;V)$ and $H_{sing}(G;V)$ coincide.*

3. Continuous Local Cochains on k-Spaces

For compactly Hausdorff generated spaces X one can use abelian k-groups V as coefficients and work in the subcategory **kTop** of compactly Hausdorff generated spaces only. The proofs presented in Section 2 carry over and one obtains analogous results for k-spaces and k-modules. In the following we work out the details. (The properties of k-spaces used here can be found in the appendix.)

Let X be a k-space and V be a k-group which is a module (in **kTop**) over a k-ring R. For each open covering $\mathfrak{U} := \{U_i \mid i \in I\}$ of X the spaces $\mathfrak{U}[q]$ are open subspaces of X^{q+1}. Because the coreflector $\mathrm{k} : \mathbf{Top} \to \mathbf{Top}$ preserves open embeddings (Lemma B.12) the corresponding k-spaces $\mathrm{k}\mathfrak{U}[q]$ form an open simplicial subspace of the simplicial k-space $\mathrm{k}X^{q+1}$; this allows us to define a cosimplicial abelian of continuous n-cochains with values in V:

$$A_{kc}(\mathfrak{U};V) := C(\mathrm{k}\mathfrak{U},V)$$

The cohomology of the associated cochain complex $A^*_{kc}(\mathfrak{U};V)$ is denoted by $H_{kc}(\mathfrak{U};V)$. Since each k-space $\mathrm{k}U_i^{q+1}$ is an open subspace of X^{q+1}, we can consider the presheafs $A^q_{kc}(-;V) := C(\mathrm{k}(-^p),V)$ and the sub double complex $\check{C}^*(\mathfrak{U},A^*_{kc})$ of $\check{C}^*(\mathfrak{U},A^*)$ of $\check{C}^*(\mathfrak{U},A^*)$. The rows of this sub double complex $\check{C}^*(\mathfrak{U},A^*_{kc})$ can be augmented by the complex $A^*_{kc}(\mathfrak{U};V)$ of continuous $\mathfrak{U}$-local cochains and the columns can be augmented by the Čech-complex $\check{C}^*(\mathfrak{U};V)$ for the covering $\mathfrak{U}$. These augmentations induce homomorphisms $i^*_{kc} : A^*_{kc}(\mathfrak{U};V) \to \mathrm{Tot}\check{C}^*(\mathfrak{U},A^*_{kc})$ and $j^*_{kc} : \check{C}^*(\mathfrak{U};V) \to \mathrm{Tot}\check{C}^*(\mathfrak{U},A^*_{kc})$ of cochain complexes respectively.

Lemma 3.1. *The homomorphism $j^*_c : \check{C}^*(\mathfrak{U};V) \to \mathrm{Tot}\check{C}^*(\mathfrak{U},A^*_c)$ induces an isomorphism in cohomology.*

Proof. The proof is analogous to that of Lemma 1.1. □

In the following the coefficients V will always be a k-module over a unital k-ring R. (Different coefficient groups are considered in Section 6.)

Proposition 3.2. *For any R-valued partition of unity $\{\varphi_{q,i} \mid i \in I\}$ subordinate to the covering $\{\mathrm{k}U_i^{q+1} \mid i \in I\}$ of $\mathrm{k}\mathfrak{U}[q]$ the homomorphisms*

$$h^{p,q} : \check{C}^p(\mathfrak{U}, A^q) \to \check{C}^{p-1}(\mathfrak{U}, A^q), \quad h^{p,q}(f)_{i_0 \dots i_{p-1}} = \sum_i \varphi_{q,i} \cdot f_{i i_0 \dots i_{p-1}} \tag{3.1}$$

form a row contraction of the augmented row $A^q(\mathfrak{U}; V) \hookrightarrow \check{C}^*(\mathfrak{U}, A^q)$ which restricts to a row contraction of the augmented sub-row $A^q_{kc}(\mathfrak{U}; V) \hookrightarrow \check{C}^*(\mathfrak{U}, A^q_{kc})$.

Proof. The proof is analogous to that of Proposition 2.4. □

Corollary 3.3. *For any open covering $\mathfrak{U} = \{U_i \mid i \in I\}$ of a k-space X for which the coverings $\{\mathrm{k}U_i^{q+1} \mid i \in I\}$ of the k-spaces $\mathfrak{U}[q]$ are R-numerable the homomorphism $i^*_{kc} : A^*_{kc}(\mathfrak{U}; V) \to \mathrm{Tot}\check{C}^*(\mathfrak{U}, A^*_{kc})$ induces an isomorphism in cohomology.*

Theorem 3.4. *For any open covering $\mathfrak{U}$ of a k-space X for which each covering $\{\mathrm{k}U_i^{q+1} \mid i \in I\}$ of $\mathrm{k}\mathfrak{U}[q]$ is R-numerable the inclusion $A^*_{kc}(\mathfrak{U}; V) \hookrightarrow A^*(\mathfrak{U}; V)$ induces an isomorphism in cohomology and the cohomologies $\check{H}(\mathfrak{U}; V)$, $H_{kc}(\mathfrak{U}; V)$ and $H(\mathfrak{U}; V)$ are isomorphic.*

Proof. The proof is analogous to that of Theorem 2.7. □

In this case the Čech Cohomology $\check{H}(\mathfrak{U}; V)$ for the covering $\mathfrak{U}$ of X can be either computed from the complex $A^*_{kc}(\mathfrak{U}; V)$ of continuous $\mathfrak{U}$-local cochains or from from the complex $A^*(\mathfrak{U}; V)$ of $\mathfrak{U}$-local cochains.

Corollary 3.5. *If the ring R is a complete k-field, $\{\varphi_i \mid i \in I\}$ a generalised R-valued partition of unity on X and $\mathfrak{U} := \{\varphi_i^{-1}(R \setminus \{0\}) \mid i \in I\}$ then the inclusion $A^*_c(\mathfrak{U}; V) \hookrightarrow A^*(\mathfrak{U}; V)$ induces an isomorphism in cohomology and the cohomologies $\check{H}(\mathfrak{U}; V)$, $H_{kc}(\mathfrak{U}; V)$ and $H(\mathfrak{U}; V)$ are isomorphic.*

If the products X^{q+1} in **Top** are already compactly Hausdorff generated, then the open subspaces $\mathfrak{U}[q]$ also are k-spaces. This in particular happens if X is a metric space, locally compact or a Hausdorff k_ω-space (cf. Lemma C.7). In this case the diagonal neighbourhoods of the form $\mathfrak{U}[q]$ are cofinal in all diagonal neighbourhoods and the colimit $\mathrm{colim}\, H_{kc}(\mathfrak{U}; V)$ over all R-numerable coverings $\mathfrak{U}$ is called the *Alexander-Spanier cohomology w.r.t R-numerable coverings.*

Corollary 3.6. *For metric, locally compact or Hausdorff k_ω-spaces X the cohomologies $\check{H}(X; V)$, $H_{AS,kc}(X; V)$ and $H_{AS}(X; V)$ w.r.t. R-numerable coverings are isomorphic.*

Example 3.7. Real and complex Kac-Moody groups are Hausdorff k_ω-spaces (cf. [GGH06]). Thus for real or complex Kac-Moody groups G the cohomologies $\check{H}(G; V)$, $H_{AS,kc}(G; V)$ and $H_{AS}(G; V)$ w.r.t. R-numerable coverings are isomorphic.

Corollary 3.8. *The cohomologies $\check{H}(X; V)$, $H_{AS,kc}(X; V)$ and $H_{AS}(X; V)$ of a metric, paracompact and locally compact or paracompact Hausdorff k_ω-space X are isomorphic.*

Example 3.9. For metrisable manifolds M and real k-modules V the cohomologies $\check{H}(M; V)$, $H_{AS,kc}(M; V)$ and $H_{AS}(M; V)$ are isomorphic.

Proposition 3.10. *If $d : X \times X \to \mathbb{R}$ is a continuous pseudometric on X and V a real k-module then for each $\epsilon > 0$ and covering $\mathfrak{U} = \{B_d(x, \epsilon) \mid x \in X\}$ of X by open ϵ-balls the inclusion $A^*_{kc}(\mathfrak{U}; V) \hookrightarrow A^*(\mathfrak{U}; V)$ induces an isomorphism in cohomology and the cohomologies $\check{H}(\mathfrak{U}; V)$, $H_{kc}(\mathfrak{U}; V)$ and $H(\mathfrak{U}; V)$ are isomorphic.*

Proof. The proof is analogous to that of Proposition 2.11. □

Example 3.11. If $\mathfrak{U} = \{B(x, \epsilon) \mid x \in X\}$ is an open covering of a finite dimensional Riemannian manifold M by open ϵ-balls and V a real k-module, then the cohomology $H_{kc}(\mathfrak{U}; V)$ of the complex $A^*_{kc}(\mathfrak{U}; V)$ is isomorphic to cohomology $H(\mathfrak{U}; V)$ and to the Čech and singular cohomologies of M (cf. Ex. 1.15). If M is an infinite dimensional Riemannian manifold one has to require the local existence of geodesics.

Corollary 3.12. *For any open entourage U of a uniform k-space X and real k-module V the inclusion $A^*_{kc}(\mathfrak{U}_U; V) \hookrightarrow A^*(\mathfrak{U}_U; V)$ induces an isomorphism in cohomology and the cohomologies $\check{H}(\mathfrak{U}_U; V)$, $H_{kc}(\mathfrak{U}_U; V)$ and $H(\mathfrak{U}_U; V)$ are isomorphic.*

A particular interesting case are compactly Hausdorff generated topological groups G. For such groups and k-modules V over the k-ring $\mathbb{R}$ we observe:

Corollary 3.13. *For any open 1-neighbourhood U of a compactly Hausdorff generated topological group G and real k-module V the inclusion $A^*_{kc}(\mathfrak{U}_U; V) \hookrightarrow A^*(\mathfrak{U}_U; V)$ induces an isomorphism in cohomology and the cohomologies $\check{H}(\mathfrak{U}_U; V)$, $H_c(\mathfrak{U}_U; V)$ and $H(\mathfrak{U}_U; V)$ are isomorphic.*

This especially applies to all metric groups, all locally compact groups and all topological groups which are Hausdorff k_ω-spaces:

Example 3.14. For any Hilbert Lie group, real or complex Kac-Moody group G and real k-module V the inclusion $A^*_{kc}(\mathfrak{U}_U; V) \hookrightarrow A^*(\mathfrak{U}_U; V)$ induces an isomorphism in cohomology and the cohomologies $\check{H}(\mathfrak{U}_U; V)$, $H_c(\mathfrak{U}_U; V)$ and $H(\mathfrak{U}_U; V)$ are isomorphic.

Proposition 3.15. *For any open covering $\mathfrak{U}$ of a k-space X for which each set $U_{i_0 \ldots i_p}$ is V-acyclic and each covering $\{\mathrm{k}U_i^{q+1} \mid i \in I\}$ of $\mathrm{k}\mathfrak{U}[q]$ is R-numerable the homomorphism $C(\lambda^*_{\mathfrak{U}}, V) : A^*_{kc}(\mathfrak{U}; V) \to S^*(X, \mathfrak{U}; V)$ induces an isomorphism $H_{kc}(\mathfrak{U}; V) \cong H_{sing}(X; V)$ in cohomology and the following diagram is commutative:*

$$\begin{tikzcd}
H_{kc}(\mathfrak{U}; V) \arrow[r, "H(i)", "\cong"'] \arrow[d, "H(C(\lambda^*_{\mathfrak{U}}; V))"', "\cong"] & H(\mathfrak{U}; V) \arrow[r, "H(j)^{-1} H(i)", "\cong"'] \arrow[d, "H(C(\lambda^*_{\mathfrak{U}}; V))"', "\cong"] & \check{H}(\mathfrak{U}; V) \arrow[d, equal] \\
H_{sing}(\mathfrak{U}; V) \arrow[r, equal] & H_{sing}(\mathfrak{U}; V) \arrow[r, "\cong", "H(j)^{-1} H(i)"'] & \check{H}(\mathfrak{U}; V)
\end{tikzcd}$$

In particular the Čech and the continuous $\mathfrak{U}$-local cohomology do not depend on the open cover $\mathfrak{U}$ subject to the above conditions chosen.

Corollary 3.16. *For any open covering $\mathfrak{U}$ of a k-space X for which each set $U_{i_0 \ldots i_p}$ is V-acyclic and each covering $\{\mathrm{k}U_i^{q+1} \mid i \in I\}$ of $\mathrm{k}\mathfrak{U}[q]$ is R-numerable the singular cohomology $H_{sing}(X; V)$ and the Čech cohomology $\check{H}(\mathfrak{U}; V)$ for the covering $\mathfrak{U}$ can be computed from the complex $A^*_{kc}(\mathfrak{U}; V)$ of continuous $\mathfrak{U}$-local cochains.*

Example 3.17. For any 'good' cover $\mathfrak{U}$ of a k-space X for which each covering $\{\mathrm{k}U_i^{q+1} \mid i \in I\}$ of $\mathrm{k}\mathfrak{U}[q]$ is R-numerable the morphism $A_{kc}^*(\mathfrak{U};V) \to S^*(X,\mathfrak{U};V)$ of cochain complexes induces an isomorphism in cohomology and the cohomologies $\check{H}(\mathfrak{U};V)$, $H(\mathfrak{U};V)$, $H_{kc}(\mathfrak{U};V)$ and $H_{sing}(X;V)$ are isomorphic.

Lemma 3.18. *If X is a k-space and the diagonal neighbourhoods $\mathfrak{U}[q]$ for open coverings $\mathfrak{U}$ for which the sets $U_{i_0\dots i_p}$ are V-acyclic and each covering $\{\mathrm{k}U_i^{q+1} \mid i \in I\}$ of $\mathrm{k}\mathfrak{U}[q]$ is R-numerable are cofinal in all diagonal neighbourhoods, then for each such covering $\mathfrak{U}$ the cohomology $H_{kc}(\mathfrak{U};V)$ of continuous $\mathfrak{U}$-local cochains coincides with the continuous Alexander-Spanier cohomology $H_{AS,kc}(X;V)$ of X. In particular the directed system $H_{kc}(\mathfrak{U};V)$ of abelian groups is co-Mittag-Leffler.*

Example 3.19. If $\mathfrak{U}$ is an open covering of a finite dimensional Riemannian manifold M by geodetically convex sets, then the cohomology of the complex $A_{kc}^*(\mathfrak{U};V)$ is isomorphic to the Čech cohomology $\check{H}(\mathfrak{U};V)$ and to the singular cohomology $H_{sing}(M;V)$ of M. If M is an infinite dimensional Riemannian manifold one has to require the local existence of geodesics for this argument to be applicable.

Example 3.20. If G is a Hilbert Lie group and U a geodetically convex identity neighbourhood of G, then the cohomology of the complex $A_{kc}^*(\mathfrak{U}_U;V)$ is isomorphic to Čech cohomology $\check{H}(\mathfrak{U};V)$ and to the singular cohomology $H_{sing}(M;V)$ of M.

One obtains a similar result for locally contractible compactly Hausdorff generated topological groups without acyclicity condition on the open coverings:

Theorem 3.21. *For any locally contractible compactly Hausdorff generated group G with open neighbourhood filterbase $\mathcal{U}_1$ for which all finite products G^{p+1} are k-spaces and any real k-module V the morphisms $A_{kc}^*(\mathfrak{U}_U;V) \hookrightarrow A^*(\mathfrak{U}_U;V)$ and $C(\lambda_{\mathfrak{U}_U}^*,V) : A_{kc}^*(\mathfrak{U}_U;V) \to S^*(\mathfrak{U}_U;V)$ for all $U \in \mathcal{U}_1$ induce isomorphisms*

$$\operatorname{colim}_{U\in\mathcal{U}_1} H_{kc}(\mathfrak{U}_U;V) \cong \operatorname{colim}_{U\in\mathcal{U}_1} H(\mathfrak{U}_U;V) \cong H_{sing}(G;V)\,.$$

in cohomology.

Corollary 3.22. *For metrisable Lie groups G with open identity neighbourhood filter $\mathcal{U}_1$ and real k-modules V the cohomologies $\operatorname{colim}_{U\in\mathcal{U}_1} H_{kc}(\mathfrak{U}_U;V)$, $\operatorname{colim}_{U\in\mathcal{U}_1} H(\mathfrak{U}_U;V)$, $\operatorname{colim}_{U\in\mathcal{U}_1} \check{H}(\mathfrak{U}_U;V)$ and $H_{sing}(G;V)$ coincide.*

4. Smooth Local Cochains

We use the differential calculus over general base fields and rings presented in [BGN04] and assume that the ring R to be a smooth manifold with smooth addition and multiplication. For an open covering $\mathfrak{U}$ of a (possibly infinite dimensional) differential manifold M and abelian Lie groups V which are smooth R-modules one can consider the complex $A_s^*(\mathfrak{U};V) = C^\infty(\mathfrak{U}[*],V)$ of smooth $\mathfrak{U}$-local cochains. Replacing the pre-sheaf $A^q(-;V)$ of q-cochains by the pre-sheaf $A_s^q(-;V) = C^\infty(-^q;V)$ of smooth q-cochains we obtain a sub double complex $\check{C}^*(\mathfrak{U},A_s^*)$ of $\check{C}^*(\mathfrak{U},A_c^*)$ whose groups are given by

$$\check{C}^p(\mathfrak{U},A_s^q) := \left\{ f \in \check{C}^p(\mathfrak{U},A^q) \mid \forall i_0,\dots i_p \in I : f_{i_0\dots i_p} \in C^\infty(U_{i_0\dots i_p}^{q+1};V) \right\}\,.$$

The rows of this sub double complex $\check{C}^*(\mathfrak{U},A_s^*)$ can be augmented by the complex $A_s^*(\mathfrak{U};V)$ of smooth $\mathfrak{U}$-local cochains and the columns can be augmented by the

Čech-complex $\check{C}^*(\mathfrak{U}; V)$ for the covering $\mathfrak{U}$. These augmentations induce homomorphisms $i_s^* : A_s^*(\mathfrak{U}; V) \to \mathrm{Tot}\check{C}^*(\mathfrak{U}, A_c^*)$ and $j_s^* : \check{C}^*(\mathfrak{U}; V) \to \mathrm{Tot}\check{C}^*(\mathfrak{U}, A_s^*)$ of cochain complexes respectively.

Lemma 4.1. *The homomorphism $j_s^* : \check{C}^*(\mathfrak{U}; V) \to \mathrm{Tot}\check{C}^*(\mathfrak{U}, A_s^*)$ induces an isomorphism in cohomology.*

Proof. The proof is analogous to that of Lemma 1.1. □

Replacing continuity by smoothness, R-paracompactness by smooth R-paracompactness in the discussion in Section 2 yields:

Proposition 4.2. *For any smooth R-valued partition of unity $\{\varphi_{q,i} \mid i \in I\}$ subordinate to the covering $\{U_i^{q+1} \mid i \in I\}$ of $\mathfrak{U}[q]$ the homomorphisms*

$$h^{p,q} : \check{C}^p(\mathfrak{U}, A^q) \to \check{C}^{p-1}(\mathfrak{U}, A^q), \quad h^{p,q}(f)_{i_0 \dots i_{p-1}} = \sum_i \varphi_{q,i} \cdot f_{ii_0 \dots i_{p-1}} \tag{4.1}$$

form a row contraction of the augmented row $A^q(\mathfrak{U}; V) \hookrightarrow \check{C}^*(\mathfrak{U}, A^q)$ which restricts to a row contraction of the augmented sub-row $A_s^q(\mathfrak{U}; V) \hookrightarrow \check{C}^*(\mathfrak{U}, A_s^q)$.

Proof. The proof is analogous to that of Proposition 2.4. □

Corollary 4.3. *For any open covering $\mathfrak{U} = \{U_i \mid i \in I\}$ of a manifold M for which the coverings $\{U_i^{q+1} \mid i \in I\}$ of the manifolds $\mathfrak{U}[q]$ are smoothly R-numerable the homomorphism $i_s^* : A_s^*(\mathfrak{U}; V) \to \mathrm{Tot}\check{C}^*(\mathfrak{U}, A_s^*)$ induces an isomorphism in cohomology.*

Theorem 4.4. *For any open covering $\mathfrak{U}$ of a manifold M for which each covering $\{U_i^{q+1} \mid i \in I\}$ of $\mathfrak{U}[q]$ is smoothly R-numerable the inclusion $A_s^*(\mathfrak{U}; V) \hookrightarrow A^*(\mathfrak{U}; V)$ induces an isomorphism in cohomology and the cohomologies $\check{H}(\mathfrak{U}; V)$, $H_s(\mathfrak{U}; V)$, $H_c(\mathfrak{U}; V)$ and $H(\mathfrak{U}; V)$ are isomorphic.*

Proof. The proof is analogous to that of Theorem 2.7. □

In this case the Čech Cohomology $\check{H}(\mathfrak{U}; V)$ for the covering $\mathfrak{U}$ of X can be either computed from the complex $A_s^*(\mathfrak{U}; V)$ of smooth $\mathfrak{U}$-local cochains, the complex $A_c^*(\mathfrak{U}; V)$ of continuous $\mathfrak{U}$-local cochains or from from the complex $A^*(\mathfrak{U}; V)$ of $\mathfrak{U}$-local cochains.

Corollary 4.5. *For any smooth R-valued partition of unity $\{\varphi_i \mid i \in I\}$ on M and $\mathfrak{U} := \{\varphi_i^{-1}(R \setminus \{0\}) \mid i \in I\}$ the inclusion $A_s^*(\mathfrak{U}; V) \hookrightarrow A^*(\mathfrak{U}; V)$ induces an isomorphism in cohomology and the cohomologies $\check{H}(\mathfrak{U}; V)$, $H_s(\mathfrak{U}; V)$, $H_c(\mathfrak{U}; V)$ and $H(\mathfrak{U}; V)$ are isomorphic.*

Proof. The proof is analogous to the proof of Corollary 2.8 where the requirement of local finiteness guarantees the smoothness of the functions φ_q. □

For the ring $R = \mathbb{R}$ of reals we obtain a more general version:

Corollary 4.6. *For $R = \mathbb{R}$, any smooth generalised partition of unity $\{\varphi_i \mid i \in I\}$ on M and $\mathfrak{U} := \{\varphi_i^{-1}(R \setminus \{0\}) \mid i \in I\}$ the inclusion $A_s^*(\mathfrak{U}; V) \hookrightarrow A^*(\mathfrak{U}; V)$ induces an isomorphism in cohomology and the cohomologies $\check{H}(\mathfrak{U}; V)$, $H_s(\mathfrak{U}; V)$, $H_c(\mathfrak{U}; V)$ and $H(\mathfrak{U}; V)$ are isomorphic.*

Proof. In view of Theorem 4.4 it suffices to show that the coverings $\{U_i^{q+1} \mid i \in I\}$ of the manifolds $\mathfrak{U}[q]$ are smoothly numerable. The smooth functions $\varphi_{i_1,\ldots,i_q}$ given by $\varphi_{i_0,\ldots,i_q}(\vec{m}) := \varphi_{i_1}(m_0)\cdots\varphi_{i_q}(m_q)$, $i_0,\ldots,i_q \in I$ form a generalised partition of unity of M^{q+1}. By Lemma A.5 there exist there exist non-negative smooth real functions $\{\varphi_{i_0,\ldots,i_q,n} \mid i_0,\ldots i_q \in I,\, n \in \mathbb{N}\}$ such that for all $n \in \mathbb{N}$

(1) the collection $\{\operatorname{supp}\varphi_{\vec{i},n} \mid \vec{i} \in I^{q+1}\}$ refines $\{U_{i_0} \times \cdots \times U_{i_q} \mid i_0,\ldots,i_q \in I\}$,
(2) the collection $\{\operatorname{supp}\varphi_{\vec{i},n} \mid \vec{i} \in I^{q+1}\}$ of supports is locally finite,

and such that that for for fixed $\vec{i} \in I^{q+1}$ the supports of $\varphi_{\vec{i},n}$, $n \in \mathbb{N}$ exhaust the open set $U_{i_0} \times \cdots \times U_{i_q}$. An application of Proposition A.6 to the set $\{\varphi_{i,\ldots,i} \mid i \in I\}$ of smooth functions shows that the covering $\{U_i^{q+1} \mid i \in I\}$ of $\mathfrak{U}[q]$ is numerable. □

The colimit $H_{AS,s}(M;V) := \operatorname{colim} H(\mathfrak{U};V)$ over all smoothly R-numerable coverings $\mathfrak{U}$ is called the *Alexander-Spanier cohomology w.r.t smoothly R-numerable coverings*. Passing to the colimit over all smoothly R-numerable covers we observe:

Corollary 4.7. *The cohomologies $\check{H}(M;V)$, $H_{AS,s}(M;V)$ and $H_{AS}(M;V)$ of a manifold M w.r.t. smoothly R-numerable coverings are isomorphic.*

Calling a manifold smoothly *R-paracompact*, if every open covering $\mathfrak{U}$ of X admits a smooth R-valued partition of unity subordinate to $\mathfrak{U}$ we also note:

Corollary 4.8. *The cohomologies $\check{H}(M;V)$, $H_{AS,s}(M;V)$ and $H_{AS}(M;V)$ of a smoothly R-paracompact manifold M are isomorphic.*

Example 4.9. If $\mathfrak{U}$ is an open covering of a Riemannian manifold M and V a real topological vector space, then the cohomology $H_s(\mathfrak{U};V)$ of the complex $A_s^*(\mathfrak{U};V)$ is isomorphic to cohomology $H(\mathfrak{U};V)$. If the open sets $U \in \mathfrak{U}$ are geodsically convex then these cohomologies are also isomorphic to the Čech and singular cohomologies of M (cf. Ex 1.15). (If M is an infinite dimensional Riemannian manifold one has to require the local existence of geodesics.)

A particular interesting case are Lie groups with open coverings of the form $\mathfrak{U}_U := \{gU \mid g \in G\}$, where U is an open identity neighbourhood in G. Here the complex $A_s^*(\mathfrak{U};V)$ is sometimes called the complex of *smooth U-local cochains.* For this special case we observe:

Corollary 4.10. *For any open identity neighbourhood U of a smoothly paracompact Lie group G and any real TVS V the inclusion $A_s^*(\mathfrak{U}_U;V) \hookrightarrow A^*(\mathfrak{U}_U;V)$ induces an isomorphism in cohomology and the cohomologies $\check{H}(\mathfrak{U}_U;V)$, $H_s(\mathfrak{U}_U;V)$, $H_c(\mathfrak{U}_U;V)$ and $H(\mathfrak{U}_U;V)$ are isomorphic.*

Combining these results with those concerning singular cohomology we observe:

Proposition 4.11. *For any open covering $\mathfrak{U}$ of a smooth manifold M for which each set $U_{i_0\ldots i_p}$ is V-acyclic and each covering $\{U_i^{q+1} \mid i \in I\}$ of $\mathfrak{U}[q]$ is smoothly R-numerable the homomorphism $A_s^*(\mathfrak{U};V) \to S^*(\mathfrak{U};V)$ induces an isomorphism $H_s(\mathfrak{U};V) \cong H_{sing}(X;V)$ in cohomology and the following diagram is commutative:*

$$\begin{tikzcd}
H_s(\mathfrak{U};V) \arrow[r, "\cong"'] \arrow[d, "H(C(\lambda^*_{\mathfrak{U}};V))"', "\cong"] & H_c(\mathfrak{U};V) \arrow[r, "\cong"'] \arrow[d, "H(C(\lambda^*_{\mathfrak{U}};V))"', "\cong"] & H(\mathfrak{U};V) \arrow[r, "H(j)^{-1}H(i)", "\cong"'] \arrow[d, "H(C(\lambda^*_{\mathfrak{U}};V))"', "\cong"] & \check{H}(\mathfrak{U};V) \arrow[d, equal] \\
H_{sing}(\mathfrak{U};V) \arrow[r, equal] & H_{sing}(\mathfrak{U};V) \arrow[r, equal] & H_{sing}(\mathfrak{U};V) \arrow[r, "\cong", "H(j)^{-1}H(i)"'] & \check{H}(\mathfrak{U};V)
\end{tikzcd}$$

In particular the Čech and the smooth $\mathfrak{U}$-local cohomology do not depend on the open cover $\mathfrak{U}$ subject to the acyclicity condition chosen.

Proof. The proof is analogous to the proof of Proposition 2.15. □

Corollary 4.12. *For any open covering $\mathfrak{U}$ of a smooth manifold M for which each set $U_{i_0\ldots i_p}$ is V-acyclic and each covering $\{U_i^{q+1} \mid i \in I\}$ of $\mathfrak{U}[q]$ is smoothly R-numerable the singular cohomology $H_{sing}(M;V)$ and the Čech cohomology $\check{H}(\mathfrak{U};V)$ can both be computed from the complex $A_s^*(\mathfrak{U};V)$ of smooth $\mathfrak{U}$-local cochains.*

Example 4.13. For any 'good' covering $\mathfrak{U} = \{U_i \mid i \in I\}$ of a manifold M for which each covering $\{U_i^{q+1} \mid i \in I\}$ of $\mathfrak{U}[q]$ is smoothly R-numerable the morphism $C(\lambda_{\mathfrak{U}}^*, V) : A_s^*(\mathfrak{U};V) \to S^*(\mathfrak{U};V)$ of cochain complexes induces an isomorphism in cohomology and the cohomologies $\check{H}(\mathfrak{U};V)$, $H_s(\mathfrak{U};V)$, $H_c(\mathfrak{U};V)$, $H(\mathfrak{U};V)$ and $H_{sing}(M;V)$ are isomorphic.

Lemma 4.14. *If the open coverings $\mathfrak{U}$ of a manifold M for which the sets $U_{i_0\ldots i_p}$ are V-acyclic and each covering $\{U_i^{q+1} \mid i \in I\}$ of $\mathfrak{U}[q]$ is smoothly R-numerable are cofinal in all open coverings, then for each such covering $\mathfrak{U}$ the the cohomology $H_s(\mathfrak{U};V)$ of smooth $\mathfrak{U}$-local cochains coincides with the smooth Alexander-Spanier cohomology $H_{AS,s}(M;V)$ of M. In particular the directed system $H_s(\mathfrak{U};V)$ of abelian groups is co-Mittag-Leffler.*

Proof. In this case the (smooth) Alexander-Spanier cohomology can be computed as the colimit over this cofinal set of open coverings $\mathfrak{U}$. Proposition 4.11 shows the isomorphisms $H_c(\mathfrak{U};V) \cong H(\mathfrak{U};V) \cong H_{sing}(M;V)$ for every covering $\mathfrak{U}$ in this cofinal set; this implies the isomorphism $H_{AS,c}(M;V) \cong H_{AS}(M;V)$ of the colimit groups. It also shows that the directed systems $H_c(\mathfrak{U};V)$ and $H(\mathfrak{U};V)$ are co-Mittag-Leffler. □

Example 4.15. If $\mathfrak{U}$ is an open covering of a finite dimensional Riemannian manifold M by geodetically convex sets, then the cohomology of the complex $A_s^*(\mathfrak{U};V)$ is isomorphic to the Čech cohomology $\check{H}(\mathfrak{U};V)$ and to the singular cohomology $H_{sing}(M;V)$ of M. If M is an infinite dimensional Riemannian manifold one has to require the local existence of geodesics and smooth partitions of unity for this argument to be applicable.

Example 4.16. If G is a Hilbert Lie group and U a geodetically convex identity neighbourhood of G, then the cohomology of the complex $A_s^*(\mathfrak{U}_U;V)$ is isomorphic to Čech cohomology $\check{H}(\mathfrak{U};V)$ and to the singular cohomology $H_{sing}(M;V)$ of M.

For Lie groups one can also consider the complex $\operatorname{colim}_{U\in\mathcal{U}_1} A_s^*(\mathfrak{U}_U;V)$, where U ranges over all open identity neighbourhoods of G. In order to obtain a result similar to that for locally contractible groups (Theorem 2.21) we require the Lie groups and their finite products to be smoothly R-paracompact. Although this does not guarantee the row-exactness of the double complex $\check{C}^*(\mathfrak{U}_U, A_c^*)$, it allows us to construct approximative row contractions $h^{*,q} : \check{C}^*(\mathfrak{U}_U, A_s^q) \to \check{C}^{*-1}(\mathfrak{U}_U, A_s^q)$. These row contractions will serve the same purpose after shrinking the open covering $\mathfrak{U}_U$ to an open covering $\mathfrak{U}_V$.

Lemma 4.17. *For any open identity neighbourhood U of a Lie group G for which all finite products are smoothly R-paracompact, there exists an open identity neighbourhood $V \subseteq U$ and smooth R-valued functions $\{\varphi_{q,g} \mid G \in G\}$ with locally finite*

supports in $(gU) \times \cdots \times (gU)$ respectively such that the restriction of each function $\varphi_q = \sum_{g \in G} \varphi_{q,g}$ to $\mathfrak{V}[q]$ is the constant function 1.

Proof. Let V and W be open identity neighbourhoods satisfying $V^{-1}V \subseteq \overline{W} \subseteq U$. We assert that the closure of $\mathfrak{V}[q]$ is contained in $\mathfrak{U}[q]$. If $\vec{x}_j$ is a net in $\mathfrak{V}[q]$ converging to a point $\vec{x}$ in G^{q+1}, then the net $x_{j,0}^{-1}\vec{x}_j$ converges to $x_0^{-1}\vec{x}$. Every point $\vec{x}_j$ is contained in some open set $(gV) \times \cdots \times (gV)$, as a consequence the point $x_{j,0}^{-1}x_{j,k}$ is contained in $(gV)^{-1}gV = V^{-1}g^{-1}gV = V^{-1}V \subset \overline{W}$ and $\vec{x}$ is contained in $(x_0\overline{W}) \times \cdots \times (x_0\overline{W})$; the latter set is contained in $\mathfrak{U}[q]$. Thus the complement $A = G^{q+1} \setminus \overline{\mathfrak{V}[q]}$ and the open sets $(gU) \times \cdots \times (gU)$ form an open covering of G^{q+1}. If the Lie group G and its finite products are smoothly R-paracompact, then there exists a smooth R-valued partition of unity $\{\psi\} \cup \{\varphi_{q,g} \mid g \in G\}$ subordinate to this open covering. Since the function ψ has support in the complement of $\mathfrak{V}[q]$ the sum $\varphi_q = \sum_{g \in G} \varphi_{q,g}$ is 1 on $\mathfrak{V}[q]$. □

For any open covering $\mathfrak{U} := \{U_i \mid i \in I\}$ of a smooth manifold M, set of smooth R-valued functions $\{\varphi_{q,i} \mid i \in I\}$ with locally finite supports contained in U_i^{q+1} respectively and smooth cochain $f \in \check{C}^p(\mathfrak{U}; A_s^q)$ the products $\varphi_{q,i} f_{ii_0 \ldots i_{p-1}}$ have supports in the open sets $U_{ii_0 \ldots i_p}^{q+1}$ respectively. Therefore they can be smoothly extended to $U_{i_0 \ldots i_{p-1}}^{q+1}$ by defining it to be zero outside $U_{ii_0 \ldots i_{p-1}}^{q+1}$. Understanding each function $\varphi_{q,i} f_{ig_0 \ldots i_{p-1}}$ to be extended this way we define an approximation to the homotopy operator in Eq. 1.2:

$$h_\varphi^{p,q} : \check{C}^p(\mathfrak{U}, A^q) \to \check{C}^{p-1}(\mathfrak{U}, A^q), \quad h_\varphi^{p,q}(f)_{i_0 \ldots i_{p-1}} = \sum_{i \in I} \varphi_{q,g} f_{ii_0 \ldots i_{p-1}}$$

The homomorphisms map continuous cochains to continuous cochains and smooth cochains to smooth cochains by construction. In addition we observe:

Lemma 4.18. *For any open covering $\mathfrak{U} := \{U_i \mid i \in I\}$ of a smooth manifold M and set of smooth R-valued functions $\{\varphi_{q,i} \mid i \in I\}$ with locally finite supports contained in U_i^{q+1} respectively the homomorphisms $h_\varphi^{p,q}$ satisfy the equation*

$$\delta h_\varphi^{p,q}(f) + h_\varphi^{p+1,q}(\delta f) = \sum_i \varphi_{q,i} f \tag{4.2}$$

for all cochains $f \in \check{C}^p(\mathfrak{U}, A^q)$.

Proof. For any cochain $f \in \check{C}^p(\mathfrak{U}; A^q)$ of bidegree (p,q) the horizontal coboundary of $h_\varphi^{p,q}(f)$ computes to

$$\begin{aligned}
(\delta h_\varphi^{p,q}(f))_{i_0 \dots i_p}(\vec{x}) &= \sum_{k=0}^{p} (-1)^k h_\varphi^{p,q}(f)_{i_0 \dots \hat{i}_k \dots i_p}(\vec{x}) \\
&= \sum_{k=0}^{p} (-1)^k \sum_i \varphi_{q,i}(\vec{x}) f_{i i_0 \dots \hat{i}_k \dots i_{p-1}}(\vec{x}) \\
&= \sum_i \varphi_{q,i}(\vec{x}) \sum_{k=0}^{p} (-1)^k f_{i i_0 \dots \hat{i}_k \dots i_{p-1}}(\vec{x}) \\
&= \sum_i \varphi_{q,i}(\vec{x}) \left[f_{i_0 \dots i_p}(\vec{x}) - (\delta f)_{i i_0 \dots i_p}(\vec{x}) \right] \\
&= \sum_i \varphi_{q,i}(\vec{x}) f_{i_0 \dots i_p}(\vec{x}) - h_\varphi^{p+1,q}(\delta f)_{i i_0 \dots i_p}(\vec{x}) \,,
\end{aligned}$$

which is the stated equality. □

Proposition 4.19. *For any open identity neighbourhood U of a Lie group G for which all finite products are smoothly R-paracompact, there exists an open identity neighbourhood $V \subseteq U$ and homomorphisms $h^{p,q} : \check{C}^p(\mathfrak{U}_U, A^q) \to \check{C}^{p-1}(\mathfrak{U}_U, A^q)$ satisfying the equation*

$$\mathrm{Res}^{p,q}_{\mathfrak{U}_V, \mathfrak{U}_U} \left[\delta h^{p,q} + h^{p+1,q} \delta \right] = \mathrm{Res}^{p,q}_{\mathfrak{U}_V, \mathfrak{U}_U}$$

and which leave the sub-rows $\check{C}^*(\mathfrak{U}_U, A_s^q)$ and $\check{C}^*(\mathfrak{U}_U, A_c^q)$ invariant. In particular the colimit double complex $\mathrm{colim}_{U \in \mathcal{U}_1} \check{C}^*(\mathfrak{U}_U; A_s^*)$ is row-exact.

Proof. For any open identity neighbourhood U of a Lie group G for which all finite products are smoothly R-paracompact Lemma 4.17 shows the existence of an open identity neighbourhood $V \subseteq U$ and smooth R-valued functions $\{\varphi_{q,g} \mid G \in G\}$ with locally finite supports in $(gU) \times \cdots \times (gU)$ respectively such that the restriction of each function $\varphi_q = \sum_{g \in G} \varphi_{q,g}$ to $\mathfrak{V}[q]$ is the constant function 1. For $h^{p,q} = h_\varphi^{p,q}$ the stated equality now follows from Lemma 4.18. □

Corollary 4.20. *For any open neighbourhood filterbase $\mathcal{U}_1$ of a Lie group G whose finite products are smoothly R-paracompact the morphisms i_s^* induce an isomorphism $\mathrm{colim}_{U \in \mathcal{U}_1} H_s(\mathfrak{U}_U; V) \cong \mathrm{colim}_{U \in \mathcal{U}_1} H(\mathrm{Tot}\check{C}^*(\mathfrak{U}_U; A_s^*))$.*

Summarising the preceding observations for Lie groups we have shown:

Theorem 4.21. *For any open neighbourhood filterbase $\mathcal{U}_1$ of a Lie group G whose finite products are smoothly R-paracompact the morphisms $A_s^*(\mathfrak{U}_U; V) \hookrightarrow A^*(\mathfrak{U}_U; V)$ and $C(\lambda^*_{\mathfrak{U}_U}, V) : A_s^*(\mathfrak{U}_U; V) \to S^*(\mathfrak{U}_U; V)$ for all $U \in \mathcal{U}_1$ induce isomorphisms*

$$\mathrm{colim}_{U \in \mathcal{U}_1} H_s(\mathfrak{U}_U; V) \cong \mathrm{colim}_{U \in \mathcal{U}_1} H(\mathfrak{U}_U; V) \cong H_{sing}(G; V)$$

in cohomology.

Proof. For every Lie group G with open identity neighbourhood filterbase $\mathcal{U}_1$ the inclusions $A_s^*(\mathfrak{U}_U; V) \hookrightarrow A^*(\mathfrak{U}_U; V)$ and $\mathrm{Tot}\check{C}^*(\mathfrak{U}_U; A_s^*) \hookrightarrow \mathrm{Tot}\check{C}^*(\mathfrak{U}_U; A^*)$ of cochain

complexes lead to the commutative diagram

$$\begin{tikzcd}
\operatorname{colim}_{U\in\mathcal{U}_1} A_s^*(\mathfrak{U}_U;V) \arrow[r, "i_s^*"] \arrow[d] & \operatorname{colim}_{U\in\mathcal{U}_1} \operatorname{Tot}\check{C}^*(\mathfrak{U}_U, A_s^*) \arrow[d] & \operatorname{colim}_{U\in\mathcal{U}_1} \check{C}(\mathfrak{U}_U;V) \arrow[l, "j_s^*"'] \arrow[d, equal] \\
\operatorname{colim}_{U\in\mathcal{U}_1} A^*(\mathfrak{U}_U;V) \arrow[r, "i^*"] & \operatorname{colim}_{U\in\mathcal{U}_1} \operatorname{Tot}\check{C}^*(\mathfrak{U}_U, A^*) & \operatorname{colim}_{U\in\mathcal{U}_1} \check{C}(\mathfrak{U}_U;V) \arrow[l, "j^*"']
\end{tikzcd}$$

of cochain complexes. The morphisms j_s^* and j^* on the right hand side induce isomorphisms in cohomology, hence the inclusions of total complexes also induce isomorphisms in cohomology. The inclusions i_s^* and i^* also induce isomorphisms in cohomology (by Corollary 4.20 and Theorem 1.17), which proves the first isomorphism. The second isomorphism also follows from Theorem 1.17). □

5. Loop Contractible Coefficients

In this section we consider a different class of coefficient groups V and derive results analogous to previously obtained ones. To obtain exact rows in the double complex $\check{C}^*(\mathfrak{U}, A_c^*)$ we again impose a restriction on the coefficient group V; however this time it is an algebraic topological one:

Definition 5.1. A (semi-)topological group G is called *loop contractible*, if there exists a contraction $\Phi : G\times I \to G$ to the identity such that $\Phi_t : G \to G, g \mapsto \Phi(g,t)$ is a homomorphism of (semi-)topological groups for all $t \in I$.

Example 5.2. Any topological vector space V is loop contractible via $\Phi(v,t) = t{\cdot}v$.

Example 5.3. The path group $PG = C((I,\{0\}),(G,\{e\}))$ of based paths of a topological group G is loop contractible via $\Phi_{PG}(\gamma,s)(t) := \gamma(st)$.

Remark 5.4. A topological group G is loop contractible if and only if the extension $\Omega G \hookrightarrow PG \twoheadrightarrow G$ is a semi-direct product: If Φ_G is a loop contraction of G, then the group homomorphism $s : G \to PG, s(g)(t) = \Phi_G(g,t)$ is a right inverse to the evaluation $\mathrm{ev}_1 : PG \to G$ at 1; conversely, if such a right inverse s exists, then the homotopy given by $\Phi_G(g,t) := \mathrm{ev}_1 \Phi_{PG}(s(g))(t)$ is a loop contraction of G.

Example 5.5. For a topological group G the geometric realisation $EG := |G^{*+1}|$ of the simplicial space G^{*+1} is a semi-topological group. As observed in [BM78] the elements in EG can be identified with the step functions $f : [0,1) \to G$ which are continuous from the right and the multiplication in EG is given by the pointwise multiplication of these step functions. The natural contraction of EG is explicitely given by

$$\Phi(f,s)(t) := \begin{cases} e & \text{if}\, t < s \\ f(t) & \text{if}\, s \leq t \end{cases}$$

(cf. the contraction in [Fuc10, Section 11.2] in [BM78, p. 214]). This is a loop contraction of the semi-topological group EG.

If the abelian topological coefficient group V is loop contractible then one can generalise the classical construction of the row contractions in Proposition 2.4. For this purpose we consider the singular semi-simplicial space $C(\Delta, V)$ of V and the vertex morphism $\lambda_V : C(\Delta,V) \to V^{*+1}$ of semi-simplicial spaces, which assigns to each singular n-simplex $\tau : \Delta^n \to V$ its ordered set of vertices $(\tau(\vec{e}_0),\ldots,\tau(\vec{e}_n))$. It has been shown in [Fuc10, Chapter 3], that there exists a right inverse $\widehat{\sigma}$ to the

vertex morphism λ_V. The construction is as follows: Let $\Phi : V \times I \to V$ be a loop contraction and consider the continuous map

$$F : V \times V \times I \to V, \quad (v_0, v_1, t) \mapsto v_0 + \Phi(v_1 - v_0, t)\,,$$

then start by setting $\hat{\sigma}_0(v)(t_0) = v$ and by inductively defining the functions $\widehat{\sigma}_n$ via

$$\widehat{\sigma}_{n+1}(\vec{v})(\vec{t}) := \begin{cases} v_0 & \text{if } t_0 = 1 \\ F\left(v_0, \widehat{\sigma}_n(v_1, \ldots, v_{n+1})\left(\frac{t_1}{1-t_0}, \ldots, \frac{t_{n+1}}{1-t_0}\right), t_0\right) & \text{if } t_0 \neq 0 \end{cases} \tag{5.1}$$

We equip the spaces $C(\Delta^n, V)$ with the compact-open topology. The crucial properties of the maps $\widehat{\sigma}_n : V^{n+1} \to (\Delta^n, V)$ we rely on are:

Proposition 5.6. *The map $\widehat{\sigma} : V^{*+1} \to C(\Delta^*, V)$ is a morphism of semi-simplicial topological spaces which is a right inverse to the vertex morphism λ_V and all the adjoint functions $\sigma_n : V^{n+1} \times \Delta^n \to V$ are continuous. In addition for all $v \in V$ the singular n-simplices $\widehat{\sigma}_n(v, \ldots, v)$ are the constant maps $\Delta^n \to \{v\}$.*

Proof. The continuity of the maps $\sigma_n : V^{n+1} \times \Delta^n \to V$ is shown in [Fuc10, Lemma 3.0.69], the fact that $\widehat{\sigma}$ is a morphism of semi-simplicial spaces is the content of [Fuc10, Lemma 3.0.70]; the proofs presented there carry over in verbatim. The last statement is a consequence of the inductive definition 5.1 of the functions $\widehat{\sigma}_n$. $\square$

Lemma 5.7. *If $\Phi : V \times I \to V$ is a loop contraction, then $\widehat{\sigma} : V^{*+1} \to C(\Delta^*, V)$ is a morphism of semi-simplicial abelian topological groups.*

Proof. It is to show that each map $\widehat{\sigma}_n : V^{n+1} \to C(\Delta^n, V)$ is a group homomorphism. This is proved by induction. The functions $\widehat{\sigma}_0$ are group homomorphisms by definition. Moreover, since Φ is a loop contraction, the function $F : V \times V \times I \to V$ is additive in $V \times V$. Now assume that the function $\widehat{\sigma}_n$ is a group homomorphism and let $\vec{v}, \vec{w} \in V^{n+2}$ be given. The singular $(n+1)$-simplex $\widehat{\sigma}_{n+1}(\vec{v} + \vec{w})$ takes the value $v_0 + w_0$ at $t_0 = 1$. For $t_0 \neq 0$ its value is given by

$$\begin{aligned}
&\widehat{\sigma}_{n+1}(\vec{v} + \vec{w})(\vec{t}) = \\
&\quad = \quad F\left(v_0 + w_0, \widehat{\sigma}_n(v_1 + w_1, \ldots, v_{n+1} + w_{n+1})\left(\frac{t_1}{1-t_0}, \ldots, \frac{t_{n+1}}{1-t_0}\right), t_0\right) \\
&\quad = \quad F\left(v_0, \widehat{\sigma}_n(v_1, \ldots, v_{n+1})\left(\frac{t_1}{1-t_0}, \ldots, \frac{t_{n+1}}{1-t_0}\right), t_0\right) \\
&\qquad\quad + F\left(w_0, \widehat{\sigma}_n(w_1, \ldots, w_{n+1})\left(\frac{t_1}{1-t_0}, \ldots, \frac{t_{n+1}}{1-t_0}\right), t_0\right) \\
&\quad = \quad \widehat{\sigma}_{n+1}(\vec{v})(\vec{t}) + \widehat{\sigma}_{n+1}(\vec{w})(\vec{t})
\end{aligned}$$

which completes the inductive step. $\square$

From now on we assume the coefficient group V to be loop contractible with loop contraction $\Phi : V \times I \to V$, which gives rise to a morphism $\widehat{\sigma} : V^{*+1} \to C(\Delta^*, V)$ of semi-simplicial abelian topological groups that is a right inverse to λ_V. The above observations enable us to replace the linear combination $\sum_i \varphi_{q,i} f_{ii_0\ldots i_p}$ of functions in Proposition 1.2 by the values of the singular n-simplices $\widehat{\sigma}_n(f_{\alpha_0 i_0\ldots i_p}, \ldots, f_{\alpha_n i_0\ldots i_p})$ at $(\varphi_{q,\alpha_0}, \ldots \varphi_{q,\alpha_n})$ for certain indices $\alpha_0, \ldots, \alpha_n \in I$. For this purpose we first observe:

Lemma 5.8. *For any partition of unity $\{\varphi_{q,i} \mid i \in I\}$ subordinate to the open cover $\{U_i^{q+1} \mid i \in I\}$ of $\mathfrak{U}[q]$ and p-cochain $f \in \check{C}^p(\mathfrak{U}, A_c^q)$ the maps*

$$U^{q+1}_{i_0\ldots i_{p-1}} \to V, \quad x \mapsto \sigma_n \circ (f_{\alpha_0 i_0 \ldots i_{p-1}}, \ldots, f_{\alpha_n i_0\ldots i_{p-1}}, \varphi_{q,\alpha_0}, \ldots, \varphi_{q,\alpha_n})(\vec{x}) \tag{5.2}$$

– where for each $x \in X$ the indices $\alpha_0 < \cdots < \alpha_n$ are those for which $\varphi^{-1}_{q,\alpha_i}(\mathbb{R} \setminus \{0\})$ contains x – are continuous.

Proof. It suffices to show that each point $\vec{x} \in \mathfrak{U}[q]$ has a neighbourhood on which the functions defined in 5.2 are continuous. This is a consequence of the fact that $\hat{\sigma} : V^{*+1} \to C(\Delta, V)$ is a morphism of semi-simplicial spaces: Since the supports of the functions $\varphi_{q,i}$ are locally finite, each point $\vec{x} \in \mathfrak{U}[q]$ has a neighbourhood W such that the set $I_W := \{\alpha' \in I \mid \varphi_{q,\alpha'}(\vec{x}) \neq 0\}$ is finite. Let $\alpha'_0 < \cdots < \alpha'_k$ be the ordered set of indices in I_W. The function defined by

$$x \mapsto \sigma_k(f_{\alpha'_0 i_0\ldots i_{p-1}}(x), \ldots, f_{\alpha'_k i_0 \ldots i_{p-1}}(x), \varphi_{q,\alpha'_0}(x), \ldots, \varphi_{q,\alpha'_k}(x))$$

is continuous on the open set $U^{q+1}_{\alpha'_0\ldots\alpha'_n i_0\ldots i_{p-1}}$, which is a neighbourhood of $\vec{x}$. Let $\alpha_0 < \cdots < \alpha_n$ be the ordered set of indices for which $\varphi^{-1}_{q,\alpha_i}(\mathbb{R} \setminus \{0\})$ contains $\vec{x}$; it is a subset of I_W. The fact that σ is a morphism of semi-simplicial spaces implies the equality

$$\begin{aligned} \sigma_k(f_{\alpha'_0 i_0\ldots i_{p-1}}(x), \ldots, f_{\alpha'_k i_0\ldots i_{p-1}}(x), \varphi_{q,\alpha'_1}(x), \ldots, \varphi_{q,\alpha'_k}(x)) = \\ \sigma_n(f_{\alpha_0 i_0\ldots i_{p-1}}(x), \ldots, f_{\alpha_n i_0 \ldots i_{p-1}}(x), \varphi_{q,\alpha_0}(x), \ldots, \varphi_{q,\alpha_n}(x)) \end{aligned}$$

which shows that the function defined in 5.2 is continuous on the open neighbourhood $U^{q+1}_{\alpha'_0\ldots\alpha'_n i_0\ldots i_{p-1}}$ of $\vec{x}$. □

Lemma 5.9. *For any partition of unity $\{\varphi_{q,i} \mid i \in I\}$ subordinate to the open cover $\{U_i^{q+1} \mid i \in I\}$ of $\mathfrak{U}[q]$ and p-cochain $f \in \check{C}^p(\mathfrak{U}, A_c^q)$ the maps defined in 5.2 form a cochain in $\check{C}^{p-1}(\mathfrak{U}; A_c^q)$.*

Proof. It is to show that for each p-cochain $f \in \check{C}^p(\mathfrak{U}, A_c^q)$ and permutation s of $\{i_0 \ldots i_p\}$ the maps defined in 5.2 satisfy the equalities

$$\begin{aligned} &\sigma_n(f_{\alpha_0 i_0 \ldots i_{p-1}}(x), \ldots, f_{\alpha_n i_0\ldots i_{p-1}}(x), \varphi_{q,\alpha_0}(x), \ldots, \varphi_{q,\alpha_n}(x)) = \\ &\quad \operatorname{sign}(s)\sigma_n(f_{\alpha_0 i_{s(0)}\ldots i_{s(p-1)}}(x), \ldots, f_{\alpha_n i_{s(0)}\ldots i_{s(p-1)}}(x), \varphi_{q,\alpha_0}(x), \ldots, \varphi_{q,\alpha_n}(x)) \,. \end{aligned}$$

This is a consequence of Lemma 5.7. Thus the assignment in 5.2 defines a cochain in $\check{C}^{p-1}(\mathfrak{U}; A_c^q)$. □

Proposition 5.10. *For any partition of unity $\{\varphi_{q,i} \mid i \in I\}$ subordinate to the open cover $\{U_i^{q+1} \mid i \in I\}$ of $\mathfrak{U}[q]$ the homomorphisms*

$$\begin{aligned} h^{p,q} : \check{C}^p(\mathfrak{U}, A^q) &\to \check{C}^{p-1}(\mathfrak{U}, A^q) \\ (h^{p,q} f)_{i_0\ldots i_{p-1}} &= \sigma_n \circ (f_{\alpha_0 i_0 \ldots i_{p-1}}, \ldots, f_{\alpha_n i_0 \ldots i_{p-1}}, \varphi_{q,\alpha_0}, \ldots, \varphi_{q,\alpha_n}) \,, \end{aligned} \tag{5.3}$$

– where for each $x \in X$ the indices $\alpha_0 < \cdots < \alpha_n$ are those satisfying $\varphi_{q,\alpha_i}(\vec{x}) \neq 0$ – form a contraction of the augmented row $A^q(\mathfrak{U}; V) \hookrightarrow \check{C}^*(\mathfrak{U}, A^q)$ which restricts to a row contraction of the augmented sub-complex $A_c^q(\mathfrak{U}; V) \hookrightarrow \check{C}^*(\mathfrak{U}, A_c^q)$.

Proof. The maps $h^{p,q}$ are homomorphisms of abelian groups by Lemma 5.7 and map the subgroups $\check{C}^*(\mathfrak{U}, A_c^q)$ of continuous cochains into each other by Lemma 5.8. Consider a point $\vec{x} \in \mathfrak{U}[q]$ and let $\alpha_0, \ldots, \alpha_n$ be the ordered set of indices in I

for which $\varphi_{q,i}^{-1}((0,1])$ contains the point $\vec{x}$. The evaluation of $(h^{p+1,q}\delta f)_{i_0\dots i_p}$ at $\vec{x}$ computes to

$$\begin{aligned}(h^{p+1}\delta f)_{i_0\dots i_p}(\vec{x}) = \sigma_n((\delta f)_{\alpha_0 i_0\dots i_p}(x),\dots,(\delta f)_{\alpha_n i_0\dots i_p}(x),\varphi_{q,\alpha_0}(x),\dots,\varphi_{q,\alpha_n}(\vec{x}))\\ = \quad & \sigma_n(f_{i_0\dots i_p}(x),\dots,f_{i_0\dots i_p}(\vec{x}),\varphi_{q,\alpha_0}(x),\dots,\varphi_{q,\alpha_n}(\vec{x}))\\ & -\sum_k(-1)^k\sigma_n(f_{\alpha_0 i_0\dots\hat{i}_k\dots i_p},\dots f_{\alpha_n i_0\dots\hat{i}_k\dots i_p},\varphi_{q,\alpha_0}(x),\dots,\varphi_{q,\alpha_n}(\vec{x}))\\ = \quad & f_{i_0\dots i_p}(x)-(\delta h^p f)_{i_0\dots i_p}(\vec{x})\,.\end{aligned}$$

Thus the homomorphisms $h^{p,q}$ form a row contraction of $A^q(\mathfrak{U};A)\hookrightarrow \check{C}^*(\mathfrak{U},A^q)$, which restricts to a row contraction of the sub-complex $A_c^q(\mathfrak{U};A_c)\hookrightarrow \check{C}^*(\mathfrak{U},A_c^q)$. □

Corollary 5.11. *For any open covering $\mathfrak{U}=\{U_i \mid i\in I\}$ of a topological space X for which the coverings $\{U_i^{q+1}\mid i\in I\}$ of the spaces $\mathfrak{U}[q]$ are numerable the homomorphism $i_c^*: A_c^*(\mathfrak{U};V)\to \mathrm{Tot}\check{C}^*(\mathfrak{U},A_c^*)$ induces an isomorphism in cohomology.*

Recalling the contractibility condition imposed on the coefficient group V we proceed to show:

Theorem 5.12. *For any loop contractible abelian topological group V and open covering $\mathfrak{U}$ of a topological space X for which each covering $\{U_i^{q+1}\mid i\in I\}$ of $\mathfrak{U}[q]$ is numerable the inclusion $A_c^*(\mathfrak{U};V)\hookrightarrow A^*(\mathfrak{U};V)$ induces an isomorphism in cohomology and the cohomologies $\check{H}(\mathfrak{U};V)$, $H_c(\mathfrak{U};V)$ and $H(\mathfrak{U};V)$ are isomorphic.*

Proof. The proof is analogous to that of Theorem 2.7. □

In this case the Čech Cohomology $\check{H}(\mathfrak{U};V)$ for the covering $\mathfrak{U}$ of X can be either computed from the complex $A_c^*(\mathfrak{U};V)$ of continuous $\mathfrak{U}$-local cochains or from from the complex $A^*(\mathfrak{U};V)$ of $\mathfrak{U}$-local cochains.

Corollary 5.13. *For any loop contractible abelian topological group V, generalised partition of unity $\{\varphi_i\mid i\in I\}$ on X and $\mathfrak{U}:=\{\varphi_i^{-1}(R\setminus\{0\})\mid i\in I\}$ the inclusion $A_c^*(\mathfrak{U};V)\hookrightarrow A^*(\mathfrak{U};V)$ induces an isomorphism in cohomology and the cohomologies $\check{H}(\mathfrak{U};V)$, $H_c(\mathfrak{U};V)$ and $H(\mathfrak{U};V)$ are isomorphic.*

Proof. The proof is analogous to that of Corollary 2.8. □

Passing to the colimit over all numerable coverings yields the classical results:

Corollary 5.14. *For any topological space X and loop contractible abelian topological group V the Čech cohomology $\check{H}(X;V)$ w.r.t. numerable coverings and the continuous Alexander-Spanier cohomology $H_{AS,c}(X;V)$ w.r.t. numerable coverings are isomorphic.*

Corollary 5.15. *For any paracompact topological space X and loop contractible coefficient group V the Čech cohomology $\check{H}(X;V)$ and the continuous Alexander-Spanier cohomology $H_{AS,c}(X;V)$ are isomorphic.*

Example 5.16. If a paracompact space X has trivial Čech cohomology $\check{H}(X;V)$ (e.g. if X is contractible) and V is loop contractible, then the continuous Alexander-Spanier cohomology $H_{AS}(X;V)$ is trivial as well.

As we did before, we apply these observations to uniform spaces X with open coverings of the form $\mathfrak{U}_U:=\{U[x]\mid x\in X\}$, where U is an open entourage of the diagonal in $X\times X$.

Proposition 5.17. *If $d : X \times X \to \mathbb{R}$ is a continuous pseudometric on X then for each $\epsilon > 0$, covering $\mathfrak{U} = \{B_d(x, \epsilon) \mid x \in X\}$ of X by open ϵ-balls and loop contractible abelian topological group V the inclusion $A_c^*(\mathfrak{U}; V) \hookrightarrow A^*(\mathfrak{U}; V)$ induces an isomorphism in cohomology and the cohomologies $\check{H}(\mathfrak{U}; V)$, $H_c(\mathfrak{U}; V)$ and $H(\mathfrak{U}; V)$ are isomorphic.*

Proof. The proof is analogous to that of Proposition 2.11. □

Example 5.18. If $\mathfrak{U} = \{B(x, \epsilon) \mid x \in X\}$ is an open covering of a finite dimensional Riemannian manifold M by open ϵ-balls and the coefficient group V is loop contractible, then the cohomology $H_c(\mathfrak{U}; V)$ of the complex $A_c^*(\mathfrak{U}; V)$ is isomorphic to cohomology $H(\mathfrak{U}; V)$ and to the Čech and singular cohomologies of M (cf. Ex 1.15). If M is an infinite dimensional Riemannian manifold one has to require the local existence of geodesics.

Corollary 5.19. *For any open entourage U of a uniform space X and loop contractible abelian topological group V the inclusion $A_c^*(\mathfrak{U}_U; V) \hookrightarrow A^*(\mathfrak{U}_U; V)$ induces an isomorphism in cohomology and the cohomologies $\check{H}(\mathfrak{U}_U; V)$, $H_c(\mathfrak{U}_U; V)$ and $H(\mathfrak{U}_U; V)$ are isomorphic.*

For topological groups with open coverings of the form $\mathfrak{U}_U := \{gU \mid g \in G\}$, where U is an open identity neighbourhood in G we observe:

Corollary 5.20. *For any open identity neighbourhood U of a topological group G and loop contractible coefficient group V the inclusion $A_c^*(\mathfrak{U}_U; V) \hookrightarrow A^*(\mathfrak{U}_U; V)$ induces an isomorphism in cohomology and the cohomologies $\check{H}(\mathfrak{U}_U; V)$, $H_c(\mathfrak{U}_U; V)$ and $H(\mathfrak{U}_U; V)$ are isomorphic.*

Combining these results with those concerning singular cohomology (obtained in Section 1) we observe:

Proposition 5.21. *For any loop contractible abelian group V and open covering $\mathfrak{U}$ of a space X for which each set $U_{i_0 \ldots i_p}$ is V-acyclic and each covering $\{U_i^{q+1} \mid i \in I\}$ of $\mathfrak{U}[q]$ is numerable the homomorphism $C(\lambda_{\mathfrak{U}}^*, V) : A_c^*(\mathfrak{U}; V) \to S^*(X, \mathfrak{U}; V)$ induces an isomorphism $H_c(\mathfrak{U}; V) \cong H_{sing}(X; V)$ in cohomology and the following diagram is commutative:*

$$\begin{tikzcd}
H_c(\mathfrak{U}; V) \arrow[r, "H(i)", "\cong"'] \arrow[d, "H(C(\lambda_{\mathfrak{U}}^*; V))"', "\cong"] & H(\mathfrak{U}; V) \arrow[r, "H(j)^{-1}H(i)", "\cong"'] \arrow[d, "H(C(\lambda_{\mathfrak{U}}^*; V))"', "\cong"] & \check{H}(\mathfrak{U}; V) \arrow[d, equal] \\
H_{sing}(\mathfrak{U}; V) \arrow[r, equal] & H_{sing}(\mathfrak{U}; V) \arrow[r, "\cong", "H(j)^{-1}H(i)"'] & \check{H}(\mathfrak{U}; V)
\end{tikzcd}$$

In particular the Čech and the continuous $\mathfrak{U}$-local cohomology do not depend on the open cover $\mathfrak{U}$ subject to the above conditions chosen.

Corollary 5.22. *For loop contractible coefficient groups V and any open covering $\mathfrak{U}$ of a topological space X for which each set $U_{i_0 \ldots i_p}$ is V-acyclic and each covering $\{U_i^{q+1} \mid i \in I\}$ of $\mathfrak{U}[q]$ is numerable the singular cohomology $H_{sing}(X; V)$ and the Čech cohomology $\check{H}(\mathfrak{U}; V)$ for the covering $\mathfrak{U}$ can be computed from the complex $A_c^*(\mathfrak{U}; V)$ of continuous $\mathfrak{U}$-local cochains.*

Example 5.23. For any loop contractible coefficient group V and 'good' cover $\mathfrak{U}$ of a topological space X for which each covering $\{U_i^{q+1} \mid i \in I\}$ of $\mathfrak{U}[q]$ is numerable the morphism $A_c^*(\mathfrak{U};V) \to S^*(X,\mathfrak{U};V)$ of cochain complexes induces an isomorphism in cohomology and the cohomologies $\check{H}(\mathfrak{U};V)$, $H(\mathfrak{U};V)$, $H_c(\mathfrak{U};V)$ and $H_{sing}(X;V)$ are isomorphic.

Lemma 5.24. *If V is loop contractible and the open coverings $\mathfrak{U}$ of a topological space X for which the sets $U_{i_0 \ldots i_p}$ are V-acyclic and each covering $\{U_i^{q+1} \mid i \in I\}$ of $\mathfrak{U}[q]$ is numerable are cofinal in all open coverings, then for each such covering $\mathfrak{U}$ the the cohomology $H_c(\mathfrak{U};V)$ of continuous $\mathfrak{U}$-local cochains coincides with the continuous Alexander-Spanier cohomology $H_{AS,c}(X;V)$ of X. In particular the directed system $H_c(\mathfrak{U};V)$ of abelian groups is co-Mittag-Leffler.*

Proof. The proof is analogous to that of Lemma 2.18. □

Example 5.25. If $\mathfrak{U}$ is an open covering of a finite dimensional Riemannian manifold M by geodetically convex sets and V is loop contractible, then the cohomology of the complex $A_c^*(\mathfrak{U};V)$ is isomorphic to the Čech cohomology $\check{H}(\mathfrak{U};V)$ and to the singular cohomology $H_{sing}(M;V)$ of M. If M is an infinite dimensional Riemannian manifold one has to require the local existence of geodesics for this argument to be applicable.

Example 5.26. If G is a Hilbert Lie group, U a geodetically convex identity neighbourhood of G and V is loop contractible, then the cohomology of the complex $A_c^*(\mathfrak{U}_U;V)$ is isomorphic to Čech cohomology $\check{H}(\mathfrak{U};V)$ and to the singular cohomology $H_{sing}(G;V)$ of G.

As observed before, one can obtain a similar result for locally contractible topological groups without acyclicity condition on the open coverings:

Theorem 5.27. *For any locally contractible group G with open identity neighbourhood filterbase $\mathcal{U}_1$ and loop contractible V the morphisms $A_c^*(\mathfrak{U}_U;V) \hookrightarrow A^*(\mathfrak{U}_U;V)$ and $C(\lambda_{\mathfrak{U}_U}^*, V) : A_c^*(\mathfrak{U}_U;V) \to S^*(\mathfrak{U}_U;V)$ for all $U \in \mathcal{U}_1$ induce isomorphisms*

$$\operatorname{colim}_{U \in \mathcal{U}_1} H_c(\mathfrak{U}_U;V) \cong \operatorname{colim}_{U \in \mathcal{U}_1} H(\mathfrak{U}_U;V) \cong H_{sing}(G;V) \,.$$

in cohomology.

Proof. The proof is analogous to that of Theorem 5.27. □

Corollary 5.28. *For Lie groups G with open identity neighbourhood filter base $\mathcal{U}_1$ and loop contractible coefficient groups V the cohomologies $\operatorname{colim}_{U \in \mathcal{U}_1} H_c(\mathfrak{U}_U;V)$, $\operatorname{colim}_{U \in \mathcal{U}_1} H(\mathfrak{U}_U;V)$, $\operatorname{colim}_{U \in \mathcal{U}_1} \check{H}(\mathfrak{U}_U;V)$ and $H_{sing}(G;V)$ coincide.*

6. k-Spaces and Loop Contractible Coefficients

In this section we work in the category of k-spaces and derive results analogous to previously obtained ones. To obtain exact rows in the double complex $\check{C}^*(\mathfrak{U}, A_{kc}^*)$ we again impose the restriction of loop contractibility.

Example 6.1. Any k-vector space V is loop contractible via $\Phi(v,t) = t \cdot v$.

Example 6.2. The path k-group $\mathrm{k}PG = \mathrm{k}C((I,\{0\}),(G,\{e\}))$ of based paths of a k-group G is loop contractible via $\Phi_{PG}(\gamma,s)(t) := \gamma(st)$.

Remark 6.3. A k-group G is loop contractible if and only if the extension $\mathrm{k}\Omega G \hookrightarrow \mathrm{k}PG \twoheadrightarrow G$ is a semi-direct product: If Φ_G is a loop contraction of G, then the group homomorphism $s : G \to \mathrm{k}PG, s(g)(t) = \Phi_G(g,t)$ is a right inverse to the evaluation $\mathrm{ev}_1 : PG \to G$ at 1; conversely, if such a right inverse s exists, then the homotopy given by $\Phi_G(g,t) := \mathrm{ev}_1 \Phi_{PG}(s(g))(t)$ is a loop contraction of G.

Example 6.4. For a k-group G the geometric realisation $\mathrm{k}EG = |\mathrm{k}G^{*+1}|$ of the simplicial k-group G^{*+1} is a k-group. As observed in [BM78] the elements in $\mathrm{k}EG$ can be identified with the step functions $f : [0,1) \to G$ which are continuous from the right and the multiplication in EG is given by the pointwise multiplication of these step functions. The natural contraction of EG is explicitely given by

$$\Phi(f,s)(t) := \begin{cases} e & \text{if}\, t < s \\ f(t) & \text{if}\, s \leq t \end{cases}$$

(cf. the contraction in [Fuc10, Section 11.2] in [BM78, p. 214]). This is a loop contraction of the k-group EG.

If the coefficient k-group V is loop contractible then one can transfer the construction of the semi-simplicial morphism $\widehat{\sigma} : V^{*+1} \to C(\mathit{\Delta}^*, V)$ constructed in Section 5 to the category of k-spaces. This is done by replacing products in **Top** by products in **kTop** in the inductive definition of $\widehat{\sigma}$. We denote the so obtained map $\mathrm{k}V^{*+1} \to \mathrm{k}C(\mathit{\Delta}, X)$ by $\mathrm{k}\widehat{\sigma}$.

Proposition 6.5. *The map $\mathrm{k}\widehat{\sigma} : \mathrm{k}V^{*+1} \to \mathrm{k}C(\mathit{\Delta}^*, V)$ is a morphism of semi-simplicial k-spaces which is a right inverse to the vertex morphism $\mathrm{k}\lambda_V$ and all the adjoint functions $\mathrm{k}\sigma_n : \mathrm{k}V^{n+1} \times \mathit{\Delta}^n \to V$ are continuous. In addition for all $v \in V$ the singular n-simplices $\widehat{\sigma}_n(v, \ldots, v)$ are the constant maps $\mathit{\Delta}^n \to \{v\}$.*

Proof. The proof is analogous to that of Proposition 5.6. □

Lemma 6.6. *If $\Phi : V \times I \to V$ is a loop contraction, then $\widehat{\sigma} : \mathrm{k}V^{*+1} \to \mathrm{k}C(\mathit{\Delta}^*, V)$ is a morphism of semi-simplicial abelian topological groups.*

Proof. The proof is analogous to that of Lemma 6.6. □

From now on we assume the coefficient k-group V to be loop contractible with loop contraction $\Phi : V \times I \to V$, which gives rise to a morphism $\mathrm{k}\widehat{\sigma} : \mathrm{k}V^{*+1} \to \mathrm{k}C(\mathit{\Delta}^*, V)$ of semi-simplicial abelian k-groups that is a right inverse to $\mathrm{k}\lambda_V$. The above observations enable us to replace the linear combination $\sum_i \varphi_{q,i} f_{ii_0 \ldots i_p}$ of functions in Proposition 3.2 by the values of the singular n-simplices $\widehat{\sigma}_n(f_{\alpha_0 i_0 \ldots i_p}, \ldots, f_{\alpha_n i_0 \ldots i_p})$ at $(\varphi_{q,\alpha_0}, \ldots \varphi_{q,\alpha_n})$ for certain indices $\alpha_0, \ldots, \alpha_n \in I$. For this purpose we first observe:

Lemma 6.7. *For any partition of unity $\{\varphi_{q,i} \mid i \in I\}$ subordinate to the open cover $\{\mathrm{k}U_i^{q+1} \mid i \in I\}$ of $\mathrm{k}\mathfrak{U}[q]$ and p-cochain $f \in \check{C}^p(\mathfrak{U}, A_{kc}^q)$ the maps*

$$\mathrm{k}U^{q+1}_{i_0 \ldots i_{p-1}} \to V, \quad x \mapsto \sigma_n \circ (f_{\alpha_0 i_0 \ldots i_{p-1}}, \ldots, f_{\alpha_n i_0 \ldots i_{p-1}}, \varphi_{q,\alpha_0}, \ldots, \varphi_{q,\alpha_n})(\vec{x}) \tag{6.1}$$

– where for each $x \in X$ the indices $\alpha_0 < \cdots < \alpha_n$ are those for which $\varphi_{q,\alpha_i}^{-1}(\mathbb{R} \setminus \{0\})$ contains x – are continuous.

Proof. The proof is analogous to that of Lemma 5.8. □

Lemma 6.8. *For any partition of unity $\{\varphi_{q,i} \mid i \in I\}$ subordinate to the open cover $\{\mathrm{k}U_i^{q+1} \mid i \in I\}$ of $\mathrm{k}\mathfrak{U}[q]$ and p-cochain $f \in \check{C}^p(\mathfrak{U}, A^q_{kc})$ the maps defined in 5.2 form a cochain in $\check{C}^{p-1}(\mathfrak{U}; A^q_{kc})$.*

Proof. The proof is analogous to that of Lemma 5.9. □

Proposition 6.9. *For any partition of unity $\{\varphi_{q,i} \mid i \in I\}$ subordinate to the open cover $\{\mathrm{k}U_i^{q+1} \mid i \in I\}$ of $\mathrm{k}\mathfrak{U}[q]$ the homomorphisms*

$$\begin{aligned} (6.2)\ h^{p,q} : \check{C}^p(\mathfrak{U}, A^q) &\to \check{C}^{p-1}(\mathfrak{U}, A^q) \\ (h^{p,q} f)_{i_0 \ldots i_{p-1}} &= \sigma_n \circ (f_{\alpha_0 i_0 \ldots i_{p-1}}, \ldots, f_{\alpha_n i_0 \ldots i_{p-1}}, \varphi_{q,\alpha_0}, \ldots, \varphi_{q,\alpha_n}), \end{aligned}$$

– where for each $x \in X$ the indices $\alpha_0 < \cdots < \alpha_n$ are those satisfying $\varphi_{q,\alpha_i}(\vec{x}) \neq 0$ – form a contraction of the augmented row $A^q(\mathfrak{U}; V) \hookrightarrow \check{C}^*(\mathfrak{U}, A^q)$ which restricts to a row contraction of the augmented sub-complex $A^q_{kc}(\mathfrak{U}; V) \hookrightarrow \check{C}^*(\mathfrak{U}, A^q_{kc})$.

Proof. The proof is analogous to that of Proposition 6.9. □

Corollary 6.10. *For any open covering $\mathfrak{U} = \{U_i \mid i \in I\}$ of a k-space X for which the coverings $\{\mathrm{k}U_i^{q+1} \mid i \in I\}$ of the k-spaces $\mathrm{k}\mathfrak{U}[q]$ are numerable the homomorphism $i^*_{kc} : A^*_{kc}(\mathfrak{U}; V) \to \mathrm{Tot}\check{C}^*(\mathfrak{U}, A^*_{kc})$ induces an isomorphism in cohomology.*

Recalling the contractibility condition imposed on the coefficient group V we proceed to show:

Theorem 6.11. *For any loop contractible abelian k-group V and open covering $\mathfrak{U}$ of a k-space X for which each covering $\{\mathrm{k}U_i^{q+1} \mid i \in I\}$ of $\mathrm{k}\mathfrak{U}[q]$ is numerable the inclusion $A^*_{kc}(\mathfrak{U}; V) \hookrightarrow A^*(\mathfrak{U}; V)$ induces an isomorphism in cohomology and the cohomologies $\check{H}(\mathfrak{U}; V)$, $H_{kc}(\mathfrak{U}; V)$ and $H(\mathfrak{U}; V)$ are isomorphic.*

Proof. The proof is analogous to that of Theorem 2.7. □

In this case the Čech Cohomology $\check{H}(\mathfrak{U}; V)$ for the covering $\mathfrak{U}$ of X can be either computed from the complex $A^*_{kc}(\mathfrak{U}; V)$ of continuous $\mathfrak{U}$-local cochains or from from the complex $A^*(\mathfrak{U}; V)$ of $\mathfrak{U}$-local cochains.

Corollary 6.12. *For any loop contractible abelian k-group V, generalised partition of unity $\{\varphi_i \mid i \in I\}$ on X and $\mathfrak{U} := \{\varphi_i^{-1}(R \setminus \{0\}) \mid i \in I\}$ the inclusion $A^*_{kc}(\mathfrak{U}; V) \hookrightarrow A^*(\mathfrak{U}; V)$ induces an isomorphism in cohomology and the cohomologies $\check{H}(\mathfrak{U}; V)$, $H_{kc}(\mathfrak{U}; V)$ and $H(\mathfrak{U}; V)$ are isomorphic.*

Proof. The proof is analogous to that of Corollary 2.8. □

Recall that for all metric, locally compact or Hausdorff k_ω-spaces X the products X^{p+1} are already compactly Hausdorff generated (cf. Lemma C.7); for these kinds of spaces the diagonal neighbourhoods of the form $\mathfrak{U}[q]$ are cofinal in all diagonal neighbourhoods.

Corollary 6.13. *For metric, locally compact or Hausdorff k_ω-spaces X and loop contractible abelian k-groups V the Čech cohomology $\check{H}(X; V)$ w.r.t. numerable coverings and the continuous Alexander-Spanier cohomology $H_{AS,kc}(X; V)$ w.r.t. numerable coverings are isomorphic.*

Example 6.14. Real and complex Kac-Moody groups are Hausdorff k_ω-spaces (cf. [GGH06]). Thus for real or complex Kac-Moody groups G and loop contractible abelian k-groups V the cohomologies $\check{H}(G; V)$, $H_{AS,kc}(G; V)$ and $H_{AS}(G; V)$ w.r.t. numerable coverings are isomorphic.

Corollary 6.15. *For any metric, paracompact and locally compact or paracompact Hausdorff k_ω-space X and loop contractible coefficient k-group V the Čech cohomology $\check{H}(X;V)$ and the continuous Alexander-Spanier cohomology $H_{AS,c}(X;V)$ are isomorphic.*

Example 6.16. If a metric, paracompact and locally compact or paracompact Hausdorff k_ω-space X has trivial Čech cohomology $\check{H}(X;V)$ (e.g. if X is contractible) and V is loop contractible, then the continuous Alexander-Spanier cohomology $H_{AS}(X;V)$ is trivial as well.

As we did before, we apply these observations to uniform k-spaces X with open coverings of the form $\mathfrak{U}_U := \{U[x] \mid x \in X\}$, where U is an open entourage of the diagonal in $X \times X$.

Proposition 6.17. *If $d : X \times X \to \mathbb{R}$ is a continuous pseudometric on X then for each $\epsilon > 0$, covering $\mathfrak{U} = \{B_d(x,\epsilon) \mid x \in X\}$ of X by open ϵ-balls and loop contractible abelian k-group V the inclusion $A^*_{kc}(\mathfrak{U};V) \hookrightarrow A^*(\mathfrak{U};V)$ induces an isomorphism in cohomology and the cohomologies $\check{H}(\mathfrak{U};V)$, $H_{kc}(\mathfrak{U};V)$ and $H(\mathfrak{U};V)$ are isomorphic.*

Proof. The proof is analogous to that of Proposition 2.11. □

Example 6.18. If $\mathfrak{U} = \{B(x,\epsilon) \mid x \in X\}$ is an open covering of a finite dimensional Riemannian manifold M by open ϵ-balls and the coefficient k-group V is loop contractible, then the cohomology $H_{kc}(\mathfrak{U};V)$ of the complex $A^*_{kc}(\mathfrak{U};V)$ is isomorphic to cohomology $H(\mathfrak{U};V)$ and to the Čech and singular cohomologies of M (cf. Ex 1.15). If M is an infinite dimensional Riemannian manifold one has to require the local existence of geodesics.

Corollary 6.19. *For any open entourage U of a uniform k-space X and loop contractible abelian k-group V the inclusion $A^*_{kc}(\mathfrak{U}_U;V) \hookrightarrow A^*(\mathfrak{U}_U;V)$ induces an isomorphism in cohomology and the cohomologies $\check{H}(\mathfrak{U}_U;V)$, $H_{kc}(\mathfrak{U}_U;V)$ and $H(\mathfrak{U}_U;V)$ are isomorphic.*

For compactly Hausdorff generated topological groups with open coverings of the form $\mathfrak{U}_U := \{gU \mid g \in G\}$, where U is an open identity neighbourhood in G we observe:

Corollary 6.20. *For any open identity neighbourhood U of a compactly Hausdorff generated topological group G and loop contractible k-group V the inclusion $A^*_{kc}(\mathfrak{U}_U;V) \hookrightarrow A^*(\mathfrak{U}_U;V)$ induces an isomorphism in cohomology and the cohomologies $\check{H}(\mathfrak{U}_U;V)$, $H_c(\mathfrak{U}_U;V)$ and $H(\mathfrak{U}_U;V)$ are isomorphic.*

Combining these results with those concerning singular cohomology (obtained in Section 1) we observe:

Proposition 6.21. *For any loop contractible abelian k-group V and open covering $\mathfrak{U}$ of a k-space X for which each set $U_{i_0\ldots i_p}$ is V-acyclic and each covering $\{\mathrm{k}U_i^{q+1} \mid i \in I\}$ of $\mathrm{k}\mathfrak{U}[q]$ is numerable the homomorphism $C(\lambda^*_{\mathfrak{U}},V) : A^*_{kc}(\mathfrak{U};V) \to S^*(X,\mathfrak{U};V)$ induces an isomorphism $H_{kc}(\mathfrak{U};V) \cong H_{sing}(X;V)$ in cohomology and*

the following diagram is commutative:

$$\begin{array}{ccccc} H_{kc}(\mathfrak{U};V) & \xrightarrow[\cong]{H(i)} & H(\mathfrak{U};V) & \xrightarrow[\cong]{H(j)^{-1}H(i)} & \check{H}(\mathfrak{U};V) \\ {\scriptstyle H(C(\lambda^*_{\mathfrak{U}};V))}\downarrow\cong & & {\scriptstyle H(C(\lambda^*_{\mathfrak{U}};V))}\downarrow\cong & & \Vert \\ H_{sing}(\mathfrak{U};V) & = & H_{sing}(\mathfrak{U};V) & \xrightarrow[H(j)^{-1}H(i)]{\cong} & \check{H}(\mathfrak{U};V) \end{array}$$

In particular the Čech and the continuous $\mathfrak{U}$-local cohomology do not depend on the open cover $\mathfrak{U}$ subject to the above conditions chosen.

Corollary 6.22. *For loop contractible coefficient k-groups V and any open covering $\mathfrak{U}$ of a k-space X for which each set $U_{i_0\ldots i_p}$ is V-acyclic and each covering $\{\mathrm{k}U_i^{q+1} \mid i \in I\}$ of $\mathrm{k}\mathfrak{U}[q]$ is numerable the singular cohomology $H_{sing}(X;V)$ and the Čech cohomology $\check{H}(\mathfrak{U};V)$ for the covering $\mathfrak{U}$ can be computed from the complex $A^*_{kc}(\mathfrak{U};V)$ of continuous $\mathfrak{U}$-local cochains.*

Example 6.23. For any loop contractible coefficient group V and 'good' cover $\mathfrak{U}$ of a topological space X for which each covering $\{U_i^{q+1} \mid i \in I\}$ of $\mathfrak{U}[q]$ is numerable the morphism $A^*_{kc}(\mathfrak{U};V) \to S^*(X,\mathfrak{U};V)$ of cochain complexes induces an isomorphism in cohomology and the cohomologies $\check{H}(\mathfrak{U};V)$, $H(\mathfrak{U};V)$, $H_c(\mathfrak{U};V)$ and $H_{sing}(X;V)$ are isomorphic.

Lemma 6.24. *If X is a k-space, V a loop contractible k-group and the diagonal neighbourhoods $\mathfrak{U}[q]$ for open coverings $\mathfrak{U}$ for which the sets $U_{i_0\ldots i_p}$ are V-acyclic and each covering $\{\mathrm{k}U_i^{q+1} \mid i \in I\}$ of $\mathrm{k}\mathfrak{U}[q]$ is numerable are cofinal in all diagonal neighbourhoods, then for each such covering $\mathfrak{U}$ the cohomology $H_{kc}(\mathfrak{U};V)$ of continuous $\mathfrak{U}$-local cochains coincides with the continuous Alexander-Spanier cohomology $H_{AS,kc}(X;V)$ of X. In particular the directed system $H_{kc}(\mathfrak{U};V)$ of abelian groups is co-Mittag-Leffler.*

Proof. The proof is analogous to that of Lemma 2.18. □

Example 6.25. If $\mathfrak{U}$ is an open covering of a finite dimensional Riemannian manifold M by geodetically convex sets and V is loop contractible, then the cohomology of the complex $A^*_{kc}(\mathfrak{U};V)$ is isomorphic to the Čech cohomology $\check{H}(\mathfrak{U};V)$ and to the singular cohomology $H_{sing}(M;V)$ of M. If M is an infinite dimensional Riemannian manifold one has to require the local existence of geodesics for this argument to be applicable.

Example 6.26. If G is a Hilbert Lie group, U a geodetically convex identity neighbourhood of G and V is loop contractible, then the cohomology of the complex $A^*_{kc}(\mathfrak{U}_U;V)$ is isomorphic to Čech cohomology $\check{H}(\mathfrak{U};V)$ and to the singular cohomology $H_{sing}(G;V)$ of G.

As observed before, one can obtain a similar result for locally contractible compactly Hausdorff generated topological groups without acyclicity condition on the open coverings:

Theorem 6.27. *For any locally contractible compactly Hausdorff generated group G with open neighbourhood filterbase $\mathcal{U}_1$ for which all finite products G^{p+1} are k-spaces and loop contractible abelian k-group V the morphisms $A^*_{kc}(\mathfrak{U}_U;V) \hookrightarrow$*

$A^*(\mathfrak{U}_U;V)$ and $C(\lambda^*_{\mathfrak{U}_U},V) : A^*_{kc}(\mathfrak{U}_U;V) \to S^*(\mathfrak{U}_U;V)$ for all $U \in \mathcal{U}_1$ induce isomorphisms

$$\operatorname{colim}_{U\in\mathcal{U}_1} H_{kc}(\mathfrak{U}_U;V) \cong \operatorname{colim}_{U\in\mathcal{U}_1} H(\mathfrak{U}_U;V) \cong H_{sing}(G;V)\,.$$

in cohomology.

Proof. The proof is analogous to that of Theorem 5.27. □

Corollary 6.28. *For metrisable Lie groups G with open identity neighbourhood filter base $\mathcal{U}_1$ and loop contractible k-groups V the cohomologies $\operatorname{colim}_{U\in\mathcal{U}_1} H_{kc}(\mathfrak{U}_U;V)$, $\operatorname{colim}_{U\in\mathcal{U}_1} H(\mathfrak{U}_U;V)$, $\operatorname{colim}_{U\in\mathcal{U}_1} \check{H}(\mathfrak{U}_U;V)$ and $H_{sing}(G;V)$ coincide.*

7. Smoothly Loop Contractible Coefficients

For an open covering $\mathfrak{U}$ of a (possibly infinite dimensional) differential manifold M and abelian Lie groups V one can consider the complex $A^*_s(\mathfrak{U};V)$ of smooth $\mathfrak{U}$-local cochains. Analogously to the procedure for continuous cochains we impose a condition on the abelian Lie group V:

Definition 7.1. A (semi-)Lie group G is called *loop contractible*, if there exists a smooth contraction $\Phi : G \times I \to G$ to the identity such that $\Phi_t : G \to G, g \mapsto \Phi(g,t)$ is a homomorphism of (semi-)Lie groups for all $t \in I$.

Example 7.2. Any topological vector space V is smoothly loop contractible via $\Phi(v,t) = t \cdot v$.

Example 7.3. The path group $PG = C((I,\{0\}),(G,\{e\}))$ of based paths of a Lie group G is smoothly loop contractible via $\Phi_{PG}(\gamma,s)(t) := \gamma(st)$.

Remark 7.4. A Lie group G is smoothly loop contractible if and only if the extension $\Omega G \hookrightarrow PG \twoheadrightarrow G$ is a semi-direct product: If Φ_G is a smooth loop contraction of G, then the Lie group homomorphism $s : G \to PG, s(g)(t) = \Phi_G(g,t)$ is a right inverse to the evaluation $\mathrm{ev}_1 : PG \to G$ at 1; conversely, if such a right inverse s exists, then the homotopy given by $\Phi_G(g,t) := \mathrm{ev}_1 \Phi_{PG}(s(g))(t)$ is a smooth loop contraction of G.

Replacing continuity by smoothness, paracompactness by smooth paracompactness and the loop contraction by a smooth loop contraction in the previous discussion yields:

Proposition 7.5. *If V is a smoothly loop contractible abelian Lie group then there exists a morphism $\widehat{\sigma} : V^{*+1} \to C(\Delta^*,V)$ of semi-simplicial Lie groups which is a right inverse to the vertex morphism λ_V and all the adjoint functions $\sigma_n : V^{n+1} \times \Delta^n \to V$ are smooth. In addition for all $v \in V$ the singular n-simplices $\widehat{\sigma}_n(v,\dots,v)$ are the constant maps $\Delta^n \to \{v\}$.*

Proof. The morphism $\widehat{\sigma} : V^{*+1} \to C(\Delta^*,V)$ is constructed by smoothing out the construction presented in the last Section. □

From now on we assume the coefficient Lie group V to be smoothly loop contractible with smooth loop contraction $\Phi : V \times I \to V$, which gives rise to a morphism $\widehat{\sigma} : V^{*+1} \to C(\Delta^*,V)$ of semi-simplicial abelian Lie groups that is a right inverse to λ_V. Similar to the continuous case we observe:

Lemma 7.6. *For any smooth partition of unity $\{\varphi_{q,i} \mid i \in I\}$ subordinate to the open cover $\{U_i^{q+1} \mid i \in I\}$ of $\mathfrak{U}[q]$ and p-cochain $f \in \check{C}^p(\mathfrak{U}, A^q)$ the maps*

$$U^{q+1}_{i_0 \dots i_{p-1}} \to V, \quad x \mapsto \sigma_n \circ (f_{\alpha_0 i_0 \dots i_{p-1}}, \dots, f_{\alpha_n i_0 \dots i_{p-1}}, \varphi_{q,\alpha_0}, \dots, \varphi_{q,\alpha_n})(\vec{x}) \tag{7.1}$$

– where for each $x \in X$ the indices $\alpha_0 < \cdots < \alpha_n$ are those for which $\varphi^{-1}_{q,\alpha_i}(\mathbb{R} \setminus \{0\})$ contains x – are smooth.

Proof. The proof is analogous to that of Lemma 5.8. □

Lemma 7.7. *For any smooth partition of unity $\{\varphi_{q,i} \mid i \in I\}$ subordinate to the open cover $\{U_i^{q+1} \mid i \in I\}$ of $\mathfrak{U}[q]$ and p-cochain $f \in \check{C}^p(\mathfrak{U}, A^q_s)$ the maps defined in 7.1 form a cochain in $\check{C}^{p-1}(\mathfrak{U}; A^q_s)$.*

Proof. The proof is analogous to that of Lemma 5.9. □

Proposition 7.8. *For any smooth partition of unity $\{\varphi_{q,i} \mid i \in I\}$ subordinate to the open cover $\{U_i^{q+1} \mid i \in I\}$ of $\mathfrak{U}[q]$ the homomorphisms*

$$\begin{aligned} h^{p,q} : \check{C}^p(\mathfrak{U}, A^q) &\to \check{C}^{p-1}(\mathfrak{U}, A^q) \\ (h^{p,q} f)_{i_0 \dots i_{p-1}} &= \sigma_n \circ (f_{\alpha_0 i_0 \dots i_{p-1}}, \dots, f_{\alpha_n i_0 \dots i_{p-1}}, \varphi_{q,\alpha_0}, \dots, \varphi_{q,\alpha_n}), \end{aligned} \tag{7.2}$$

– where for each $x \in X$ the indices $\alpha_0 < \cdots < \alpha_n$ are those satisfying $\varphi_{q,\alpha_i}(\vec{x}) \neq 0$ – form a contraction of the augmented row $A^q(\mathfrak{U}; V) \hookrightarrow \check{C}^*(\mathfrak{U}, A^q)$ which restricts to a row contraction of the augmented sub-complex $A^q_s(\mathfrak{U}; V) \hookrightarrow \check{C}^*(\mathfrak{U}, A^q_s)$.

Proof. The proof is analogous to that of Proposition 5.10. □

Corollary 7.9. *For any open covering $\mathfrak{U} = \{U_i \mid i \in I\}$ of a manifold M for which the coverings $\{U_i^{q+1} \mid i \in I\}$ of the spaces $\mathfrak{U}[q]$ are smoothly numerable the homomorphism $i_c^* : A_s^*(\mathfrak{U}; V) \to \mathrm{Tot}\check{C}^*(\mathfrak{U}, A_s^*)$ induces an isomorphism in cohomology.*

Recalling the contractibility condition imposed on the abelian Lie group V we proceed to show:

Theorem 7.10. *For any smoothly loop contractible abelian Lie group V and open covering $\mathfrak{U}$ of a manifold M for which each covering $\{U_i^{q+1} \mid i \in I\}$ of $\mathfrak{U}[q]$ is smoothly numerable the inclusion $A_s^*(\mathfrak{U}; V) \hookrightarrow A^*(\mathfrak{U}; V)$ induces an isomorphism in cohomology and the cohomologies $\check{H}(\mathfrak{U}; V)$, $H_s(\mathfrak{U}; V)$, $H_c(\mathfrak{U}; V)$ and $H(\mathfrak{U}; V)$ are isomorphic.*

Proof. The proof is analogous to that of Theorem 2.7. □

In this case the Čech Cohomology $\check{H}(\mathfrak{U}; V)$ for the covering $\mathfrak{U}$ of X can be either computed from the complex $A_s^*(\mathfrak{U}; V)$ of smooth $\mathfrak{U}$-local cochains, from the complex $A_c^*(\mathfrak{U}; V)$ of continuous $\mathfrak{U}$-local cochains or from from the complex $A^*(\mathfrak{U}; V)$ of $\mathfrak{U}$-local cochains.

In order to obtain results similar to those for continuous cochains we require the manifold M and their finite products to be smoothly paracompact. Then passing to the colimit over all smoothly numerable coverings yields the classical results:

Corollary 7.11. *For any manifold M for which all finite powers are smoothly paracompact and any smoothly loop contractible abelian Lie group V the Čech cohomology $\check{H}(M; V)$ w.r.t. smoothly numerable coverings and the smooth Alexander-Spanier cohomology $H_{AS,s}(X; V)$ w.r.t. numerable coverings are isomorphic.*

Corollary 7.12. *For any manifold M for which all finite powers are smoothly paracompact and any and smoothly loop contractible abelian Lie group V the Čech cohomology $\check{H}(X;V)$ and the smooth Alexander-Spanier cohomology $H_{AS,s}(X;V)$ are isomorphic.*

Example 7.13. If a manifold M with smoothly paracompact powers has trivial Čech cohomology $\check{H}(M;V)$ (e.g. if M is contractible) and V is smoothly loop contractible, then the smooth Alexander-Spanier cohomology $H_{AS,s}(M;V)$ is trivial as well.

Proposition 7.14. *For any smoothly loop contractible abelian lie group V and open identity neighbourhood U of a Lie group G for which all finite products are smoothly paracompact, there exists an open identity neighbourhood $W \subseteq U$ and homomorphisms $h^{p,q} : \check{C}^p(\mathfrak{U}_U, A^q) \to \check{C}^{p-1}(\mathfrak{U}_U, A^q)$ satisfying the equation*

$$\mathrm{Res}^{p,q}_{\mathfrak{U}_W,\mathfrak{U}_U}\left[\delta h^{p,q} + h^{p+1,q}\delta\right] = \mathrm{Res}^{p,q}_{\mathfrak{U}_W,\mathfrak{U}_U}$$

and which leave the sub-rows $\check{C}^*(\mathfrak{U}_U, A^q_s)$ and $\check{C}^*(\mathfrak{U}_U, A^q_c)$ invariant. In particular the colimit double complex $\mathrm{colim}_{U\in\mathcal{U}_1} \check{C}^*(\mathfrak{U}_U; A^*_s)$ is row-exact.

Proof. For any open identity neighbourhood U of a Lie group G for which all finite products are smoothly paracompact Lemma 4.17 shows the existence of an open identity neighbourhood $W \subseteq U$ and smooth real-valued functions $\{\varphi_{q,g} \mid G \in G\}$ with locally finite supports in $(gU)\times\cdots\times(gU)$ respectively such that the restriction of each function $\varphi_q = \sum_{g\in G} \varphi_{q,g}$ to $\mathfrak{W}[q]$ is the constant function 1. Then the homomorphisms

$$(7.3)\quad \begin{aligned} h^{p,q} : \check{C}^p(\mathfrak{U}, A^q) &\to \check{C}^{p-1}(\mathfrak{U}, A^q) \\ (h^{p,q}f)_{i_0\ldots i_{p-1}} &= \sigma_n \circ \left(f_{\alpha_0 i_0\ldots i_{p-1}}, \ldots, f_{\alpha_n i_0\ldots i_{p-1}}, \varphi_{q,\alpha_0}, \ldots, \varphi_{q,\alpha_n}\right), \end{aligned}$$

– where for each $x \in X$ the indices $\alpha_0 < \cdots < \alpha_n$ are those satisfying $\varphi_{q,\alpha_i}(\vec{x}) \neq 0$ – have the desired property. □

Corollary 7.15. *For any open neighbourhood filterbase $\mathcal{U}_1$ of a Lie group G whose finite products are smoothly paracompact and smoothly loop contractible abelian Lie group V the morphisms i^*_s induce an isomorphism $\mathrm{colim}_{U\in\mathcal{U}_1} H_s(\mathfrak{U}_U;V) \cong \mathrm{colim}_{U\in\mathcal{U}_1} H(\mathrm{Tot}\check{C}^*(\mathfrak{U}_U; A^*_s))$.*

Summarising the preceding observations for Lie groups we have shown:

Theorem 7.16. *For any open neighbourhood filterbase $\mathcal{U}_1$ of a Lie group G whose finite products are smoothly paracompact and smoothly loop contractible abelian Lie group V the morphisms $A^*_s(\mathfrak{U}_U;V) \hookrightarrow A^*(\mathfrak{U}_U;V)$ and $C(\lambda^*_{\mathfrak{U}_U}, V) : A^*_s(\mathfrak{U}_U;V) \to S^*(\mathfrak{U}_U;V)$ for all $U \in \mathcal{U}_1$ induce isomorphisms*

$$\mathrm{colim}_{U\in\mathcal{U}_1} H_s(\mathfrak{U}_U;V) \cong \mathrm{colim}_{U\in\mathcal{U}_1} H(\mathfrak{U}_U;V) \cong H_{sing}(G;V)$$

in cohomology.

Proof. The proof is analogous to that of Theorem 4.21. □

Appendix A. Partitions of Unity

Lemma A.1. *For each summable set of function $\varphi_i : X \to V$, $i \in I$ into a complete Hausdorff abelian group V all subsets of functions are also summable.*

Proof. Let $\varphi_i : X \to V$, $i \in I$ be a summable set of functions into a complete abelian Hausdorff group V and $J \subseteq I$ be a subset of I. We claim that for each $x \in X$ the net of finite partial sums of $\varphi_j(x)$, $j \in J$ is a Cauchy net. For each identity neighbourhood U in V there exists a finite subset $I_{x,U}$ of I such that

$$\sum_{i \in I'} \varphi_i(x) \in \varphi(x) + U \tag{A.1}$$

for all finite supersets $I' \supset I_{x,U}$. Choose an identity neighbourhood W in V satisfying $W - W \subseteq U$ and consider the finite subset $J_{x,W} := J \cap I_{x,W}$ of J. For all finite supersets J', J'' of $J_{x,W}$ the above relation A.1 implies

$$\sum_{j \in J'} \varphi_j(x) - \sum_{j \in J''} \varphi_j(x) = \sum_{j \in J' \cup (I_{x,W} \setminus J)} \varphi_j(x) - \sum_{j \in J'' \cup (I_{x,W} \setminus J)} \varphi_j(x) \in W - W \subseteq U$$

hence the the net of finite partial sums of $\varphi_j(x)$, $j \in J$ is a Cauchy net. Since V is complete and Hausdorff, this Cauchy net converges and the limit is unique. □

Lemma A.2. *For every summable set of real valued functions $\varphi_i : X \to \mathbb{R}$, $i \in I$ the set $|\varphi_i|$, $i \in I$ of non-negative functions is also summable.*

Proof. Let φ_i, $i \in I$ be a summable set of real functions. For each point $x \in X$ split the index set I into $I_{x,+} := \{i \in I \mid \varphi_i(x) \geq 0\}$ and $I_{x,-} := \{i \in I \mid \varphi_i(x) < 0\}$. The sums $\sum_{i \in I_{x,+}} \varphi_i(x)$ and $\sum_{i \in I_{x,-}} \varphi_i(x)$ exist by Lemma A.1, hence $|\varphi_i(x)|$ is summable with sum $\sum_{i \in I} |\varphi_i(x)| = \sum_{i \in I_{x,+}} \varphi_i(x) - \sum_{i \in I_{x,-}} \varphi_i(x)$. □

Lemma A.3. *For every summable set of real valued functions $\varphi_i : X \to \mathbb{R}$, $i \in I$ with continuous sum the sum $\sum_{i \in I} |\varphi_i|$ is continuous as well. If φ_i, $i \in I$ is a (generalised) partition of unity, then $|\varphi_i| / \sum |\varphi_i|$, $i \in I$ is a non-negative (generalised) partition of unity.*

Proof. Let $\varphi_i : X \to \mathbb{R}$, $i \in I$ be a summable set of real valued functions with continuous sum φ and let ψ denote the sum of absolute values $|\varphi_i|$. The convergence $\sum \varphi_i = \varphi$ means that for all $x \in X$ and $\epsilon > 0$ there exists a finite subset $I_{m,\epsilon} \subseteq I$ such that for all supersets $I' \supseteq I_{m,\epsilon}$ the inequality

$$\left| \sum_{i \in I'} \varphi_i(x) - \varphi(x) \right| < \epsilon$$

is satisfied. The set $V_{x,\epsilon} := \{x' \in X \mid |\sum_{i \in I_{x,\epsilon}} \varphi_i(x') - \varphi(x')| < \epsilon\}$ is an open neighbourhood of x. For every $x' \in V_{x,\epsilon}$ and finite subset $J \subset I \setminus I_{x,\epsilon}$ the absolute value of the sum $\sum_{i \in J} \varphi_i(x')$ is less than 2ϵ, which implies that the sum $\sum_{i \notin I_{x,\epsilon}} |\varphi_i|$ is less than 4ϵ. The intersection

$$W_{x,\epsilon} := V_{x,\epsilon} \cap \left(\left[\sum_{i \in I_{x,\epsilon}} |\varphi_i| \right] - \psi(x) \right)^{-1} ((-4\epsilon, 4\epsilon)) \tag{A.2}$$

is an even smaller open neighbourhood of x. For all points $x' \in W_{x,\epsilon}$ we observe

$$|\psi(x') - \psi(x)| \leq \sum_{i \notin I_{x,\epsilon}} |\varphi_i|(x') + \left| \sum_{i \in I_{x,\epsilon}} |\varphi_i|(x') - \psi(x) \right| \leq 4\epsilon + 4\epsilon = 8\epsilon\,.$$

Thus for every point $x \in X$ and $\epsilon > 0$ there exists a neighbourhood W of x such that $\psi(W) \subseteq B_\epsilon(\psi(x))$, i.e. ψ is continuous. □

Lemma A.4. *For each summable set of function $\varphi_i : X \to V$, $i \in I$ with continuous sum φ into a complete Hausdorff abelian group V the sum of any subset of functions is also continuous.*

Proof. Let $\varphi_i : X \to V$, $i \in I$ be a summable set of functions into a complete abelian Hausdorff group V with continuous sum $\varphi := \sum \varphi_i$ and $J \subseteq I$ be a subset of I. We show that the sum $\varphi_J := \sum_{j \in J} \varphi_j$ is continuous at each point $x \in X$. For each identity neighbourhood U in V there exists a finite subset $I_{x,U}$ of I such that A.1 holds for all supersets $I' \supseteq I_{x,U}$. Furthermore the set

(A.3)

$$V_{x,U} := \left(\sum_{i \in I_{x,U} \setminus J} \varphi_i - \varphi_i(x) \right)^{-1} (U) \cap \left(\sum_{i \in I_{x,U}} \varphi_i - \varphi \right)^{-1} (U) \cap \cap \left(\varphi - \varphi(x)\right)^{-1} (U)$$

is an open neighbourhood of x. Choose an identity neighbourhood W in V satisfying $W + W + W - W - W \subseteq U$. For all points $x' \in V_{x,W}$ and finite supersets J' of $J_{x,W} := I_{x,W} \cap J$ we observe

$$\begin{aligned} \sum_{j \in J'} \varphi_j(x') - \sum_{j \in J'} \varphi_j(x) &= \sum_{j \in J' \cup (I_{x,W} \setminus J)} \varphi_j(x') - \sum_{j \in J' \cup (I_{x,W} \setminus J)} \varphi_j(x) \\ &\quad - \sum_{j \in (I_{x,W} \setminus J)} \varphi_j(x') + \sum_{j \in (I_{x,W} \setminus J)} \varphi_j(x) \\ &\in (\varphi(x') + W) - (\varphi(x) + W) - W \\ &\subseteq W + W - W - W \end{aligned}$$

Passage to the limit shows that the difference $\varphi_J(x') - \varphi_J(x)$ is contained in the closure of $W+W-W-W$, which in turn is contained in $W+W+W-W-W \subseteq U$. Thus for each point $x \in X$ and identity neighbourhood U of V there exists a neighbourhood $V_{x,W}$ of x such that $\varphi_J(x') - \varphi_J(x) \in U$ for all $x' \in V_{x,W}$, i.e. the function φ_J is continuous at all points $x \in X$. $\square$

Similar to continuous partitions of unity (as done in [tD91]) it can be shown that coverings by cozero sets of generalised partitions of unity are always smoothly numerable. For this purpose we will use the smooth function

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases} \tag{A.4}$$

to adapt the proof for continuous functions in [tD91] to the general smooth context.

Lemma A.5. *For every generalised smooth partition of unity $\{\varphi_i \mid i \in I\}$ on a manifold M and covering by the cozero sets $U_i := \varphi_i^{-1}(\mathbb{R} \setminus \{0\})$ there exist non-negative smooth real functions $\{\varphi_{i,n} \mid i \in I,\, n \in \mathbb{N}\}$ such that for all $n \in \mathbb{N}$*

1. *the collection of supports $\{\operatorname{supp} \varphi_{i,n} \mid i \in I\}$ refines $\{U_i \mid i \in I\}$,*
2. *the collection $\{\operatorname{supp} \varphi_{i,n} \mid i \in I\}$ of supports is locally finite,*

and such that that for every point $m \in M$ some $\varphi_{i,n}$ satisfies $\varphi_{i,n}(m) > 0$ and for fixed $i \in I$ the supports of $\varphi_{i,n}$, $n \in \mathbb{N}$ exhaust the open set U_i.

Proof. Let $\{\varphi_i \mid i \in I\}$ be a smooth generalised partition of unity on a manifold M. We define real valued functions $\varphi_{i,n}$ on M via

$$\varphi_{i,n} := f \circ \left(\varphi_i^2 - \frac{1}{(n+1)^2} \right) .$$

The support of $\varphi_{i,n}$ is contained in the cozero set $\varphi_{i,n+1}^{-1}((0,\infty))$ and for fixed $i \in I$ the supports of $\varphi_{i,n}$ $n \in \mathbb{N}$ exhaust U_i by construction; this proves (1) and the last claim. Furthermore for every point $m \in M$ the convergence $\sum \varphi_i(m) = 1$ means that for all $\epsilon > 0$ there exists a finite subset $I_{m,\epsilon} \subseteq I$ such that for all supersets $I' \supseteq I_{m,\epsilon}$ the inequality

$$\left| \sum_{i \in I'} \varphi_i(m) - 1 \right| < \epsilon$$

is satisfied. For every n the set $V_{m,n} := \{m' \in M \mid |\sum_{i \in I_{m,1/2n}} (m') - 1| < 1/2n\}$ is an open neighbourhood of m on which all functions φ_i, $i \notin I_{m,1/n}$ satisfy $|\varphi_i| < 1/n$. This implies $\varphi_i^2 \leq 1/n^2$ and $\varphi_{i,n} = 0$ on $V_{m,n}$ for all i which are not contained in the finite set $I_{m,1/2n}$. Therefore the collection $\{\varphi_{i,n+1}^{-1}((0,\infty)) \mid i \in I\}$ of cozero sets and the collection $\{\operatorname{supp} \varphi_{i,n} \mid i \in I\}$ of supports are locally finite, which proves (2). Finally, for each point $m \in M$ there exists some φ_i satisfying $\varphi_i(m) \neq 0$. For all $n \in \mathbb{N}$ satisfying $1/n < |\varphi_i(m)|$ the function $\varphi_{i,n}$ also satisfies $\varphi_{i,n}(m) \neq 0$. □

Proposition A.6. *An open covering $\mathfrak{U}$ of a smooth manifold M is smoothly numerable if and only if there exist non-negative smooth real functions $\varphi_{i,n}$, $i \in I$, $n \in \mathbb{N}$ such that for all $n \in \mathbb{N}$*

1. *the collection of supports $\{\operatorname{supp} \varphi_{i,n} \mid i \in I\}$ refines $\mathfrak{U}$,*
2. *the collection $\{\operatorname{supp} \varphi_{i,n} \mid i \in I\}$ of supports is locally finite,*

and such that that for every point $m \in M$ some $\varphi_{i,n}$ satisfies $\varphi_{i,n}(m) > 0$.

Proof. The proof is an adaption of the proof of [tD91, Lemmata 5.5, 4.6] to the general smooth context. The forward implication is proved in Lemma A.5, so only the backward implication requires proof. Let $\varphi_{i,n}$, $i \in I$, $n \in \mathbb{N}$ be smooth real valued functions with the above properties. Replacing $\varphi_{in,n}$ with $\varphi_{i,n}^2/(1+\varphi_{i,n}^2)$ we can w.l.o.g. assume that the functions $\varphi_{i,n}$ take values in the uni interval. For each $n \in \mathbb{N}$ the collection $\{\operatorname{supp} \varphi_{i,k} \mid i \in I,\ k < n\}$ of supports is locally finite, hence the sum

$$q_n := \sum_{i \in I, k < n} \varphi_{i,k}$$

(where $q_0 = 0$) is a smooth real valued function on M. As a consequence the composition $\psi_{i,n} := f \circ (\varphi_{i,n} - n \cdot q_n)$ is smooth as well. We claim that the collection $\{\psi_{i,n}^{-1}(0,\infty) \mid i \in I,\ n \in \mathbb{N}\}$ of cozero sets an open covering of M and that the supports $\{\operatorname{supp} \psi_{i,n} \mid i \in I,\ n \in \mathbb{N}\}$ form a locally finite covering of M which refines the open covering $\mathfrak{U}$. If n is minimal such there exists a function $\varphi_{i,n}$ satisfying $\varphi_{i,n}(m) \neq 0$, then $q_n(m) = 0$ and $\psi_{i,n}(m) = \varphi_{i,n} - 0 \neq 0$. Thus for each point $m \in M$ there exists an index (i,n) such that $\psi_{i,n}(m) \neq 0$, i.e. the collection $\{\psi_{i,n}^{-1}((0,\infty)) \mid i \in I,\ n \in \mathbb{N}\}$ of cozero sets is an open covering of M. Moreover the sequence of functions q_n is monotone increasing, so for each $m \in M$ there exists $N \in \mathbb{N}$ such that $N \cdot q_N(m) > 1$ hence also $N \cdot q_N(m) > 1$ in a neighbourhood V_m of m. As a consequence the supports of all the functions $\psi_{i,n}$ with $n > N$ do not intersect V_m. So the collection $\{\operatorname{supp} \psi_{i,n} \mid i \in I,\ n \in \mathbb{N}\}$ of supports is locally finite. Normalisation of $\{\psi_{i,n} \mid i \in I, n \in \mathbb{N}\}$ yields a smooth partition of unity subordinate to $\mathfrak{U}$. □

Theorem A.7. *Every covering of a manifold M by the cozero sets of a generalised partition of unity $\{\varphi_i \mid i \in I\}$ is numerable.*

Proof. Let $\{\varphi_i \mid i \in I\}$ be a generalised smooth partition of unity on a manifold M and construct non-negative functions $\varphi_{i,n}$ as in the proof of Lemma A.5 and further a partition of unity $\{\psi_{i,n} \mid i \in I,\, n \in \mathbb{N}\}$ as in Proposition A.6. Then $\{\psi_i \mid i \in I,\, n \in \mathbb{N}\}$ is a partition of unity subordinate to $\{\varphi_i^{-1}(\mathbb{R} \setminus 0) \mid i \in I\}$. □

Appendix B. k-Spaces

Definition B.1. A continuous function from a compact Hausdorff space into an arbitrary topological space X is called a *probe over* X.

Definition B.2. The *k-topology* on a topological space X is the final topology of all probes over X. The underlying set of X equipped with the k-topology is denoted by $\mathrm{k}X$.

Lemma B.3. *The k-topology on a topological space X is finer than the original topology of X, i.e. the set theoretic identity map $\mathrm{k}X \to X$ is continuous.*

Definition B.4. A topological space is called a k-spaces if $\mathrm{k}X = X$. The full subcategory of **Top** with objects all k-spaces is denoted by **kTop**.

Any continuous function $f : X \to Y$ between topological space X and Y gives rise to a continuous function $\mathrm{k}(f) : \mathrm{k}X \to \mathrm{k}Y$, which coincides with f on the underlying set. These assignments constitute a functor

$$\mathrm{k} : \mathbf{Top} \to \mathbf{kTop}.$$

Proposition B.5. *The category* **kTop** *is a coreflective subcategory of* **Top** *with coreflector* k*, i.e. the functor* k *is a right adjoint to the inclusion* **kTop** → **Top**.

Proof. Let X be a k-space and Y be an arbitrary topological space. The continuous function $\epsilon_Y : \mathrm{k}Y \to Y$ induces an injective map

$$i_* : \hom_{\mathbf{kTop}}(X, \mathrm{k}Y) \to \hom_{\mathbf{Top}}(X, Y)$$

which is natural in X and Y. It remains to show that i_* is surjective. Let $f : X \to Y$ be a continuous function. Then the function $\mathrm{k}(f) : X = \mathrm{k}X \to \mathrm{k}Y$ is continuous and $i_*\mathrm{k}(f) = f$. Therefore i_* is bijective. □

Corollary B.6. *The inclusion* **kTop** → **Top** *is cocontinuous, i.e. the colimits of k-spaces in* **kTop** *coincide with those in* **Top**.

Corollary B.7. *The coreflector* $\mathrm{k} : \mathbf{Top} \to \mathbf{kTop}$ *is continuous, i.e. it preserves limits.*

Proposition B.8. *The product of spaces Y_i in* **kTop** *is given by* $\mathrm{k}\left(\prod Y_i\right)$*, where* $\prod Y_i$ *is the product in* **Top**.

Proof. By the use of Lemma B.5 one obtains for every k-space X the following chain of natural isomorphisms:

$$\begin{aligned} \hom_{\mathbf{kTop}}\left(X, \mathrm{k}\prod Y_i\right) &= \hom_{\mathbf{Top}}\left(X, \prod Y_i\right) \\ &\cong \prod \hom_{\mathbf{Top}}(X, Y_i) \cong \prod \hom_{\mathbf{kTop}}(X, Y_i), \end{aligned}$$

which is what was to be proved. □

Theorem B.9. *closed subspaces of k-spaces are k-spaces.*

Proof. Let X be a k-space and $A \subset X$ be a closed subspace. It is to show that $\mathrm{k}A \to A$ is an isomorphism, i.e. every subset B of A that is closed in $\mathbf{kTop}A$ is already closed in A. Let B be such a closed subset of $\mathrm{k}A$. Since A is closed in X, the inverse image $p^{-1}(A)$ under any probe $p : C \to X$ is closed in the compact Hausdorff space C, hence the inverse image $p^{-1}(A)$ also is compact. The restriction $p_{|p^{-1}(A)}$ of p to the compact subspace $p^{-1}(A)$ of C is a probe on A. The inverse image $p^{-1}(B)$ is equal to the inverse image $p_{|p^{-1}(A)}^{-1}(B)$ under the probe $p_{|p^{-1}(A)}$. The latter inverse image is closed in $p^{-1}(A)$ by assumption. Being a closed subspace of a closed subspace $p^{-1}(A)$ of C it is closed in C. Thus the inverse image of B under any probe $p : C \to X$ is closed in C, i.e. B is closed in X. $\square$

Lemma B.10. *The coreflector* $\mathrm{k} : \mathbf{Top} \to \mathbf{kTop}$ *preserves closed embeddings.*

Proof. Let A be a closed subspace of a topological space X and let $i_A : A \hookrightarrow X$ denote the inclusion. To show that $\mathrm{k}(i_A) : \mathrm{k}A \to \mathrm{k}X$ is closed, it suffices to prove that the image $\mathrm{k}(i_A)(B)$ of every closed subset B of $\mathrm{k}A$ is closed in $\mathrm{k}X$. If $B \subseteq \mathrm{k}A$ is such a closed subset, then every inverse image ${f'}^{-1}(B)$ under a probe $f' : C' \to A$ from a compact Hausdorff space C' into A is closed. If $f : C \to X$ is a probe, then the inverse image $C' := f^{-1}(A)$ is closed in C, hence a compact subspace of C. Therefore the restriction and corestriction $f_{|C'}^{|A}$ is a probe into A; The inverse image $f^{-1}(B)$ of B under the probe $f : C \to X$ coincides with the inverse image ${f_{|C'}^{|A}}^{-1}(B)$ of B under the probe $f_{|C'}^{|A}$, which is closed in C'. Because C' is a closed subspace of C, the inverse image $f^{-1}(B)$ also is closed in C. Thus the inverse image $f^{-1}(B)$ of B under any probe $f : C \to X$ is closed, hence B is closed in $\mathrm{k}X$. $\square$

Lemma B.11. *Any open subspace of a k-space is a k-space.*

Proof. See [tD91, Satz 6.6] for a proof. $\square$

Lemma B.12. *The coreflector* $\mathrm{k} : \mathbf{Top} \to \mathbf{kTop}$ *preserves open embeddings.*

Proof. Let U be an open subspace of a topological space X and let $j_U : U \hookrightarrow X$ denote the inclusion. For the natural transformation $\epsilon : \mathrm{k} \to \mathrm{id}_{\mathbf{Top}}$ one obtains the commutative diagram

$$\begin{tikzcd} \mathrm{k}U \arrow[r, "\mathrm{k}(j_U)"] \arrow[d, "\epsilon_U"'] & \mathrm{k}X \arrow[d, "\epsilon_X"] \\ U \arrow[r, "j_U"] & X \end{tikzcd}$$

The open subspace $\epsilon_X^{-1}(U)$ of the k-space $\mathrm{k}X$ is a k-space itself. Restricting the morphism ϵ_X to the k-space $\epsilon_X^{-1}(U)$ of $\mathrm{k}X$ and corestricting it to U we obtain the equality $\mathrm{id}_U = {\epsilon_X}_{|\epsilon_X^{-1}(U)}^{|U} \circ \mathrm{k}(j_U)$, i.e. $\mathrm{k}U$ is homeomorphic to $\epsilon_X^{-1}(U)$ and $\mathrm{k}(j_U)$ is the inclusion of an open subspace. $\square$

Appendix C. k_ω-Spaces

There exists a variant of the category $\mathbf{kTop}$ of k-spaces, which shares many of the properties of the category $\mathbf{kTop}$. It is formed by the class of all topological spaces whose topology is the weak topology with respect to some countable set of compact Hausdorff spaces:

Definition C.1. A topological space X is called a *k_ω-space* if it has the weak topology w.r.t. a countable set $\{f_n(K_n)\}$ of images of compact Hausdorff spaces K_n under continuous functions $f_n : K_n \to X$. The full subcategory of **Top** consisting with objects all k_ω-spaces is denoted by $\mathbf{k}_\omega\mathbf{Top}$.

Since the images $f_n(K_n)$ in the above definition are compact subspaces of the topological space X, this implies that X has the colimit topology of the ascending sequence $\bigcup_{i=1}^n K_i$ of compact subspaces of X. Conversely, if X is the direct limit of an ascending sequence of compact subspaces, then it is a k_ω-space. Similar to the class of k-spaces the class of k_ω-spaces can be described as quotients of compact Hausdorff-spaces:

Lemma C.2. *A topological space is a k_ω-space if and only if it is the quotient of a disjoint union of at most countably many compact Hausdorff spaces.*

Proof. The proof of [Dug89, Theorem XI.9.4] generalises to k_ω-spaces. □

Lemma C.3. *Quotients of k_ω-spaces are k_ω-spaces.*

Proof. Since compositions of quotient maps are quotient maps, this is a consequence of the characterisation of k_ω-spaces in Lemma C.2. □

Lemma C.4. *Finite and countable disjoint unions of k_ω-spaces are k_ω-spaces.*

Proof. This follows from the characterisation of k_ω-spaces in Lemma C.2. □

Summarising the last two lemmata we observe that the subcategory $\mathbf{k}_\omega\mathbf{Top}$ of **Top** has the same countable colimits:

Proposition C.5. *Finite and countable colimits of k_ω-spaces in* **Top** *are k_ω-spaces.*

Definition C.6. A topological space X is called *locally k_ω* if every point has a neighbourhood filter basis of k_ω-spaces.

Lemma C.7. *Finite products of Hausdorff k_ω-spaces are Hausdorff k_ω-spaces.*

Proof. This is a special case of [GGH06, Lemma 1.1 (b)], cf. [GGH06, Proposition 4.2 (c)]. □

References

[BGN04] W. Bertram, H. Glöckner, and K.-H. Neeb, *Differential calculus over general base fields and rings*, Expo. Math. **22** (2004), no. 3, 213–282. MR MR2069671 (2005e:26042)

[BM78] Ronald Brown and Sidney A. Morris, *Embeddings in contractible or compact objects*, Colloq. Math. **38** (1977/78), no. 2, 213–222. MR MR0578534 (58 #28249)

[Dug89] James Dugundji, *Topology*, 2 ed., Wm. C. Brown Publishers, Dubuque, Iowa, 1989, Reprinting of the 1966 original, Allyn and Bacon Series in Advanced Mathematics.

[Fuc10] Martin Fuchssteiner, *Tranformation groups and (co)homology*, Logos Verlag Berlin, 2010.

[GGH06] H. Glöckner, R. Gramlich, and T. Hartnick, *Final group topologies, Kac-Moody groups and Pontryagin duality*, ArXiv Mathematics e-prints (2006).

[Seg70] Graeme Segal, *Cohomology of topological groups*, Symposia Mathematica, Vol. IV (INDAM, Rome, 1968/69), Academic Press, London, 1970, pp. 377–387. MR MR0280572 (43 #6292)

[tD91] Tammo tom Dieck, *Topologie*, de Gruyter Lehrbuch. [de Gruyter Textbook], Walter de Gruyter & Co., Berlin, 1991. MR MR1150244 (93f:55001)

[vE62] W. T. van Est, *Local and global groups. II*, Nederl. Akad. Wetensch. Proc. Ser. A 65 = Indag. Math. **24** (1962), 409–425. MR MR0145000 (26 #2539b)

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